Sampling methods for reconstructing the geometry of a local perturbation in unknown periodic layers

Houssem Haddar, Thi-Phong Nguyen

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Abstract

This paper is dedicated to the design and analysis of sampling methods to reconstruct the shape of a local perturbation in a periodic layer from measurements of scattered waves at a fixed frequency. We first introduce the model problem that corresponds with the semi-discretized version of the continous model with respect to the Floquet-Bloch variable. We then present the inverse problem setting where (propagative and evanescent) plane waves are used to illuminate the structure and measurements of the scattered wave at a parallel plane to the periodicity directions are performed. We introduce the near field operator and analyze two possible factorizations of this operator. We then establish sampling methods to identify the defect and the periodic background geometry from this operator measurement. We also show how one can recover the geometry of the background independently from the defect. We then introduce and analyze the single Floquet-Bloch mode measurement operators and show how one can exploit them to built an indicator function of the defect independently from the background geometry. Numerical validating results are provided for simple and complex backgrounds.

Keywords: Inverse Scattering problems, Linear Sampling Method, Factorization Method, Periodic layers, Floquet-Bloch Transform

1 Introduction

We investigate in this paper the inverse problem where one is interested in reconstructing the support of a perturbation of the periodic layer from measurements of scattered waves at a fixed frequency. We are primarily concerned with the design of a sampling method that furnishes the support of the inhomogeneities without reconstructing the index of refraction. The development of sampling methods has gained a large interest in recent years and many methods have been introduced in the literature to deal with a variety of problems. We refer to [10, 11, 15, 25] for an account of recent developments of these methods. The case of periodic media has been treated by several authors and without being exhaustive, we refer to [1,2,17,22,30,31,33,36]. For the inverse problem in locally perturbed periodic waveguides we refer to [8,34] and references therein.

Up to our knowledge, the sampling methods for locally perturbed infinite periodic layers have not been treated in the literature. Even thought this problem is the one that motivates our study, we shall consider here a slightly different problem that will be referred to as the $ML-$periodic problem: it corresponds with a locally perturbed infinite periodic layer with period L that has been reduced to a domain of size ML (with M a sufficiently large parameter) with periodic boundary conditions. This is mainly due to technical reasons since our analysis for the newly introduced differential imaging functional heavily rely on the discrete Floquet-Bloch transform. According to [20] the $ML-$periodic problem can be seen as

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the discretized problem with respect to the Floquet-Bloch variable using M discretization points in the trapezoidal rule.

As an inversion method, we shall employ the recently introduced Generalized version of the Linear Sampling Method GLSM (see [4, 6, 11]) and also consider the Factorization method (see [23–25]). We consider the case where the data correspond with Rayleigh sequences of the scattered field associated with propagative and evanescent incident plane waves. For similar inverse problems in waveguides we refer to [8, 9, 30, 34]. We shall prove in a first part how the GLSM and also the factorization method can be applied to our setting.

The main contribution of our work is the design of a new sampling method that enables the imaging of the defect location without reconstructing the L periodic background. This method is in the spirit of the so-called Differential LSM that has been introduced in [5] for the imaging of defects in complex backgrounds using differential measurements. However, in our case we shall introduce a method that does not require the measurement operator for the background media. We exploit the L periodicity of the background and the Floquet-Bloch transform to design a differential criterion between different periods. This criterion is based on the study of sampling methods for the $ML-$periodic media where a single Floquet-Bloch mode is used. This study constitutes the main theoretical ingredient for our method. The sampling operator for a single Floquet-Bloch mode somehow plays the role of the measurement operator for the background media. Indeed the main interest for this new sampling method is that it is capable of identifying the defect even thought classical sampling methods fail in obtaining high fidelity reconstructions of the (complex) background media.

For this first study we shall only consider the scattering problem modeled by the Helmholtz equation. The performance of the introduced sampling methods are tested in space dimension 2 against synthetic data generated by the solver developed in [20].

The outline of this paper is as follows. We introduce in Section 2 the forward $ML-$periodic scattering problem and briefly outline the formulation of the Rayleigh radiation condition and the variational formulation of the problem. We present in Section 3 the setting of the inverse problem for incident plane waves and measurements constituted by the Rayleigh coefficients of the scattered waves. We then introduce the near field operator as well as the factorizations of this operator that will be needed for the sampling methods. Some key properties of these operators are then proved as preparation for the analysis of sampling methods. Section 4 is dedicated to the theoretical analysis of sampling methods for retrieving the geometry of the background media and the defect. We also explain in this section how these methods can be used to identify the L-periodic background media. Section 5 is dedicated to the analysis of sampling methods using a single Floquet-Bloch mode. This analysis is the last main ingredient for the differential imaging functional presented in Section 5.3. In order to make this work self-contained we included in Appendix A a summary of the main abstract theoretical results that are used for the foundations of the sampling methods.

2 Setting of the direct scattering problem

Consider a parameter $L := (L_1, \cdots, L_{d-1}) \in \mathbb{R}^{d-1},\ L_j > 0,\ j = 1, \cdots, d-1$ that will refer to the periodicity of the media with respect to the first $d-1$ variables and $M := (M_1, \cdots, M_{d-1}) \in \mathbb{N}^{d-1}$ that will refer to the number of periods in the truncated domain. A function defined in $\mathbb{R}^d$ is called L periodic if it is periodic with period L with respect to the $d-1$ first variables. We consider in the following the $ML-$periodic Helmholtz equation (vector multiplications is to be understood component wise, i.e. $ML = (M_1L_1, \cdots, M_{d-1}L_{d-1})$). In this problem, the total field satisfies

$$\begin{cases} \Delta u + k^2 n u = 0 \quad \text{in } \mathbb{R}^d,\ d = 2, 3 \\ u \text{ is } ML-\text{periodic} \end{cases} \tag{1}$$

where the *wavenumber* k is positive and real valued. We assume that the index of refraction $n \in L^\infty(\mathbb{R}^d)$ has a non negative imaginary part and is $ML-$periodic. Moreover, we assume that $n = n_p$ outside a compact domain ω where $n_p \in L^\infty(\mathbb{R}^d)$ is L-periodic and assume in addition that there exists $h > 0$ such that $n = 1$

for $|x_d| > h$ (see Fig. 1).
Thanks to the $ML-$periodicity, solving equation (1) in $\mathbb{R}^d$ is equivalent to solving it in the period

$$\Omega_M := \bigcup_{m=[-\frac{M}{2}]+1}^{m=[\frac{M}{2}]} \Omega_m = [\![M_L^-, M_L^+]\!] \times \mathbb{R}$$

where $M_L^- := \left(\left[-\frac{M}{2}\right]+\frac{1}{2}\right)L$, $M_L^+ := \left(\left[\frac{M}{2}\right]+\frac{1}{2}\right)L$ and $\Omega_m := [\![-\frac{L}{2}+mL, \frac{L}{2}+mL]\!] \times \mathbb{R}$. We here use the notation $[\![a,b]\!] := [a_1,b_1] \times \cdots \times [a_{d-1}, b_{d-1}]$ and $[\cdot]$ to denote the floor function. We denote by D (respectively D_p) a bounded domain composed by simply connected components and such that $n = 1$ (respectively $n_p = 1$) outside D (respectively D_p) and $D = D_p \cup \overline{\omega}$.

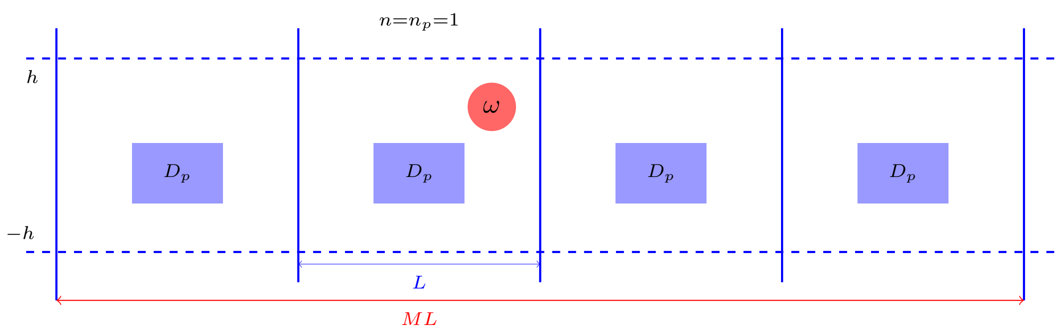

Figure 1: Sketch of the geometry for the $ML-$periodic problem

We shall consider down-to-up or up-to-down incident plane waves of the form:

$$u^i(x) = e^{\mathrm{i}\alpha_\#(j)\overline{x} \pm \mathrm{i}\beta_\#(j)x_d}$$

where

$$\alpha_\#(j) := \mathrm{i}\tfrac{2\pi}{ML}j \quad \text{and} \quad \beta_\#(j) := \sqrt{k^2 - \alpha_\#^2(j)}, \quad \mathrm{Im}\,(\beta_\#(j)) \geq 0, \quad j \in \mathbb{Z}^{d-1}$$

and where $x = (\overline{x}, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$. Then the scattered field $u^s = u - u^i$ verifies

$$\begin{cases} \Delta u^s + k^2 n u^s = -k^2(n-1)u^i \quad \text{in } \mathbb{R}^d, \\ u^s \text{ is } ML-\text{periodic} \end{cases} \tag{2}$$

and we impose as a radiation condition the Rayleigh expansions:

$$\begin{cases} u^s(\overline{x}, x_d) = \sum_{\ell \in \mathbb{Z}^{d-1}} \widehat{u^s}^+(\ell) e^{\mathrm{i}(\alpha_\#(\ell)\overline{x} + \beta_\#(\ell)(x_d - h))}, \quad \forall\ x_d > h \\ u^s(\overline{x}, x_d) = \sum_{\ell \in \mathbb{Z}^{d-1}} \widehat{u^s}^-(\ell) e^{\mathrm{i}(\alpha_\#(\ell)\overline{x} - \beta_\#(\ell)(x_d + h))}, \quad \forall\ x_d < -h, \end{cases} \tag{3}$$

where the Rayleigh coefficients $\widehat{u^s}^\pm(\ell)$ are given by

$$\begin{aligned} \widehat{u^s}^+(\ell) &:= \frac{1}{|[\![M_L^-, M_L^+]\!]|} \int_{[\![M_L^-, M_L^+]\!]} u^s(\overline{x}, h) e^{-\mathrm{i}\alpha_\#(\ell)\cdot\overline{x}}\, \mathrm{d}\overline{x} \\ \widehat{u^s}^-(\ell) &:= \frac{1}{|[\![M_L^-, M_L^+]\!]|} \int_{[\![M_L^-, M_L^+]\!]} u^s(\overline{x}, -h) e^{-\mathrm{i}\alpha_\#(\ell)\cdot\overline{x}}\, \mathrm{d}\overline{x}\,. \end{aligned} \tag{4}$$

We shall use the notation

$$\Omega_M^h := [\![M_L^-, M_L^+]\!] \times]-h, h[$$

$$\Gamma_M^h := [\![M_L^-, M_L^+]\!] \times \{h\}, \quad \Gamma_M^{-h} := [\![M_L^-, M_L^+]\!] \times \{-h\}.$$

For integer m, we denote by $H^m_\#(\Omega_M^h)$ the restrictions to Ω_M^h of functions that are in $H^m_{\text{loc}}(|x_d| \le h)$ and are $ML-$periodic. The space $H^{1/2}_\#(\Gamma_M^h)$ is then defined as the space of traces on Γ_M^h of functions in $H^1_\#(\Omega_M^h)$ and the space $H^{-1/2}_\#(\Gamma_M^h)$ is defined as the dual of $H^{1/2}_\#(\Gamma_M^h)$. Similar definitions are used for $H^{\pm 1/2}_\#(\Gamma_M^{-h})$. Using the radiation condition (3) we can define the Dirichlet-to-Neumann operators $T^\pm$ as

$$\begin{aligned}
T^+ : H^{1/2}_\#(\Gamma_M^h) &\longrightarrow H^{-1/2}_\#(\Gamma_M^h) \\
\phi &\longmapsto T^+\phi = \mathrm{i} \sum_{\ell \in \mathbb{Z}^{d-1}} \beta_\#(\ell) \widehat{\phi}^+(\ell) e^{\mathrm{i}\alpha_\#(\ell)\cdot\overline{x}} \\
T^- : H^{1/2}_\#(\Gamma_M^{-h}) &\longrightarrow H^{-1/2}_\#(\Gamma_M^{-h}) \\
\phi &\longmapsto T^-\phi = \mathrm{i} \sum_{\ell \in \mathbb{Z}^{d-1}} \beta_\#(\ell) \widehat{\phi}^-(\ell) e^{\mathrm{i}\alpha_\#(\ell)\cdot\overline{x}}
\end{aligned} \tag{5}$$

It is easy to check that $T^\pm$ are bounded operators and:

$$\operatorname{Im}\langle T^\pm\phi, \phi\rangle \ge 0, \quad \operatorname{Re}\langle T^\pm\phi, \phi\rangle \le 0, \tag{6}$$

where $\langle\cdot,\cdot\rangle$ denotes the sesquilinear duality products $H^{-1/2}_\#(\Gamma_M^h) - H^{1/2}_\#(\Gamma_M^h)$ and $H^{-1/2}_\#(\Gamma_M^{-h}) - H^{1/2}_\#(\Gamma_M^{-h})$. The scattering problem can be reformulated as: Find $u^s \in H^2_\#(\Omega_M^h)$ such that

$$\begin{aligned}
\Delta u^s + k^2 n u^s &= -k^2(n-1)u^i && \text{in} \quad \Omega_M^h \\
\frac{\partial u^s}{\partial x_d} &= \pm T^\pm(u^s) && \text{for} \ \ x_d = \pm h
\end{aligned} \tag{7}$$

and u^s is extended to Ω_M using (3). Multiplying equation (7) with $v \in H^1_\#(\Omega_M^h)$ and using the Green theorem we arrive at the variational formulation of the problem as

$$\int_{\Omega_M^h} \nabla u^s \nabla \overline{v} - k^2 n u^s \overline{v}\,\mathrm{d}x \, - \langle T^+(u^s), v\rangle - \langle T^-(u^s), v\rangle = k^2 \int_{\Omega_M^h} (n-1)u^i\overline{v}\,\mathrm{d}x \tag{8}$$

for all $v \in H^1_\#(\Omega_M^h)$. Problem (8) is of Fredholm type since the sesquilinear form

$$\mathcal{A}(u^s, v) := \int_{\Omega_M^h} \nabla u^s \nabla \overline{v} - k^2 n u^s \overline{v}\,\mathrm{d}x \, - \langle T^+(u^s), v\rangle - \langle T^-(u^s), v\rangle$$

is continuous on $H^1_\#(\Omega_M^h) \times H^1_\#(\Omega_M^h)$ and satisfies the Garding inequality

$$|\mathcal{A}(u,u)| \ge \|u\|^2_{H^1(\Omega_M^h)} - \int_{\Omega_M^h} (k^2 \operatorname{Re} n + 1)|u|^2\,\mathrm{d}x \tag{9}$$

which follows from (6). The uniqueness of solutions to this problem can be studied using Rellich type identities under some monotonicity conditions on the refractive index or by imposing that the maginary part of the refractive index is positive in an open ball (see for instance Chapter 1 of [32]).
For the purpose of this paper we shall assume that the index of refraction n is such that Problem (8) is well posed. More precisely, let $f \in L^2(\Omega_M^h)$ and consider the following variational problem: Find $w \in H^1_\#(\Omega_M^h)$ such that for all $v \in H^1_\#(\Omega_M^h)$,

$$\int_{\Omega_M^h} \nabla w \cdot \nabla \overline{v} - k^2 n w \overline{v}\,\mathrm{d}x \, - \langle T^+(w), v\rangle - \langle T^-(w), v\rangle = k^2 \int_{\Omega_M^h} (n-1) f \overline{v}\,\mathrm{d}x\,. \tag{10}$$

Then we make the following assumption:

Assumption 2.1. *We assume that n and k are such that problem* (10) *is well posed for all $f \in L^2(\Omega_M^h)$.*

We remark that the solution $w \in H^1_\#(\Omega_M^h)$ of (10) can be extended to a function in Ω_M satisfying $\Delta w + k^2 n w = -k^2(n-1)f$, using the Rayleigh expansion (3). Let G_M be the $ML-$periodic Green function satisfying $\Delta G_M + k^2 G_M = -\delta$ in Ω_M and the Rayleigh radiation condition (3). Then w can also be represented as

$$w(x) = k^2 \int_D G_M(x-y)(n-1)(f+w)(y)\,\mathrm{d}y\,. \tag{11}$$

This implies in particular that $w \in H^2_{\#,\mathrm{loc}}(\Omega_M)$, i.e. $w \in H^2_\#(\Omega^h)$ for all $h > 0$. In all the following we shall assume that the wavenumber k is such that $\beta_\#(\ell) \neq 0$ for all $\ell \in \mathbb{Z}^{d-1}$, i.e. it does not correspond with a Wood anomaly. In that case G_M has the representation

$$G_M(x) = \frac{\mathrm{i}}{2[\![ML]\!]} \sum_{\ell\in\mathbb{Z}^{d-1}} \frac{1}{\beta_\#(\ell)} e^{\mathrm{i}\alpha_\#(\ell)\overline{x} + \mathrm{i}\beta_\#(\ell)|x_d|}, \tag{12}$$

where $[\![ML]\!] := M_1 L_1 \cdots M_{d-1} L_{d-1}$.

3 Setting of the inverse problem

We first use as incident waves all down-to-up (scaled) incident plane waves $u^{i,+}(x;j)$ defined as

$$u^{i,+}(x;j) = \frac{-\mathrm{i}}{2\,\overline{\beta}_\#(j)} e^{\mathrm{i}\alpha_\#(j)\overline{x} + \mathrm{i}\overline{\beta}_\#(j)(x_d - h)},\ j \in \mathbb{Z}^{d-1}. \tag{13}$$

Then our measurements (data for the inverse problem) will be formed by the Rayleigh sequences (see (4))

$$\widehat{u^s}^+(\ell;j), \quad (j,\ell) \in \mathbb{Z}^{d-1} \times \mathbb{Z}^{d-1}$$

where j is related to the incident wave index and ℓ is related to the Rayleigh sequence index. We can also use as incident waves all up-to-down (scaled) incident plane waves $u^{i,-}(x;j)$ defined as

$$u^{i,-}(x;j) = \frac{-\mathrm{i}}{2\,\overline{\beta}_\#(j)} e^{\mathrm{i}\alpha_\#(j)\overline{x} - \mathrm{i}\overline{\beta}_\#(j)(x_d + h)},\ j \in \mathbb{Z}^{d-1}, \tag{14}$$

and as measurements (data for the inverse problem) the Rayleigh sequences (see (4))

$$\widehat{u^s}^-(\ell;j), \quad (j,\ell) \in \mathbb{Z}^{d-1} \times \mathbb{Z}^{d-1}.$$

3.1 Definition of the sampling operator

Let us consider the (Herglotz) operators $\mathcal{H}^+ : \ell^2(\mathbb{Z}^{d-1}) \to L^2(D)$ and $\mathcal{H}^- : \ell^2(\mathbb{Z}^{d-1}) \to L^2(D)$ defined by

$$\mathcal{H}^\pm a := \sum_{j\in\mathbb{Z}^{d-1}} a(j) u^{i,\pm}(\cdot;j)\big|_D, \quad \forall\, a = \{a(j)\}_{j\in\mathbb{Z}^{d-1}} \in \ell^2(\mathbb{Z}^{d-1}). \tag{15}$$

Then $\mathcal{H}^\pm$ is compact, injective (will be proved later) and its adjoint $(\mathcal{H}^\pm)^* : L^2(D) \to \ell^2(\mathbb{Z}^{d-1})$ is given by

$$(\mathcal{H}^\pm)^*\varphi := \{\widehat{\varphi}^\pm(j)\}_{j\in\mathbb{Z}^{d-1}}, \quad \text{where} \quad \widehat{\varphi}^\pm_j := \int_D \varphi(x)\overline{u^{i,\pm}(\cdot;j)}(x)\,\mathrm{d}x\,. \tag{16}$$

Let us denote by $H^\pm_{\mathrm{inc}}(D)$ the closure of the range of $\mathcal{H}^\pm$ in $L^2(D)$. We then consider the (compact) operator $\mathrm{G}^\pm : H^\pm_{\mathrm{inc}}(D) \to \ell^2(\mathbb{Z}^{d-1})$ defined by

$$\mathrm{G}^\pm(f) := \{\widehat{w}^\pm(\ell)\}_{\ell\in\mathbb{Z}^{d-1}}, \tag{17}$$

where $\{\widehat{w}^{\pm}(\ell)\}_{\ell\in\mathbb{Z}^{d-1}}$ is the Rayleigh sequence of $w \in H^1_{\#}(\Omega^h_M)$ the solution of (10). We now define the sampling operators $\mathrm{N}^{\pm} : \ell^2(\mathbb{Z}^{d-1}) \to \ell^2(\mathbb{Z}^{d-1})$ by

$$\mathrm{N}^{\pm}(a) = \mathrm{G}^{\pm}\,\mathcal{H}^{\pm}(a). \tag{18}$$

By linearity of the operators $\mathrm{G}^{\pm}$ and $\mathcal{H}^{\pm}$ we also get an equivalent definition of $\mathrm{N}^{\pm}$ as

$$[\mathrm{N}^{\pm}(a)]_{\ell} = \sum_{j\in\mathbb{Z}^{d-1}} a(j)\,\widehat{u^s}^{\pm}(\ell;j) \quad \ell\in\mathbb{Z}^{d-1}. \tag{19}$$

Let us introduce the operator $\mathrm{T} : L^2(D) \to L^2(D)$ defined by

$$\mathrm{T}f := k^2(n-1)(f + w|_D) \tag{20}$$

with w being the solution of (10). We then have the following:

Lemma 3.1. *The operators* $\mathrm{G}^{\pm}$ *defined by* (17) *can be factorized as*

$$\mathrm{G}^{\pm} = (\mathcal{H}^{\pm})^*\mathrm{T}.$$

Proof. Let $f \in L^2(D)$ and $w \in H^1_{\#}(\Omega^h_M)$ be solution to (10). By definition of the Rayleigh coefficients and combining with (12) we have

$$\begin{aligned}\widehat{w}^{\pm}(j) &= \frac{\mathrm{i}}{2[\![ML]\!]}\int_{x_d=\pm h} e^{-\mathrm{i}\alpha_{\#}(j)\overline{x}} \int_D \sum_{\ell\in\mathbb{Z}^{d-1}} \frac{1}{\beta_{\#}(\ell)} e^{\mathrm{i}\alpha_{\#}(\ell)(\overline{x}-\overline{y}) + \mathrm{i}\beta_{\#}(\ell)|h\mp y_d|}\mathrm{T}(f)(y)\,\mathrm{d}y\,\mathrm{d}\overline{x} \\ &= \int_D \frac{\mathrm{i}e^{\mathrm{i}\beta_{\#}(j)h}}{2\,\beta_{\#}(j)} e^{-\mathrm{i}\alpha_{\#}(j)\overline{y}\,\mp\,\mathrm{i}\beta_{\#}(j)y_d} Tf(y)\,\mathrm{d}y \end{aligned} \tag{21}$$

Observing that $\frac{\mathrm{i}e^{\mathrm{i}\beta_{\#}(j)h}}{2\,\beta_{\#}(j)} e^{-\mathrm{i}\alpha_{\#}(j)y_1\,\mp\,\mathrm{i}\beta_{\#}(j)y_2} = \overline{u^{i,\pm}}(y;j)$, we then have

$$\widehat{w}^{\pm}(j) = \int_D \mathrm{T}f(y)\overline{u^{i,\pm}}(y;j)\,\mathrm{d}y\,,$$

which proves the lemma. ∎

Using Lemma 3.1 we end up with

$$\mathrm{N}^{\pm} = (\mathcal{H}^{\pm})^*\,\mathrm{T}\,\mathcal{H}^{\pm}. \tag{22}$$

The justification of the Sampling Methods that will be introduced later uses the solvability of the so-called *interior transmission problem* defined as: Seek $(u,v) \in L^2(D)\times L^2(D)$ such that $u - v \in H^2(D)$ and

$$\begin{cases} \Delta u + k^2 n u = 0 & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ u - v = \varphi & \text{on } \partial D, \\ \partial(u-v)/\partial\nu = \psi & \text{on } \partial D, \end{cases} \tag{23}$$

for given $(\varphi,\psi) \in H^{3/2}(\partial D)\times H^{1/2}(\partial D)$ where ν denotes the outward normal on ∂D. Values of k for which this problem is not well posed are referred to as *transmission eigenvalues.* For a detailed discussion on the solution to this problem we refer to [11–13, 35]. For our purpose we shall assume that this problem is well posed.

Assumption 3.2. *We assume that the refractive index n and the real wave number k are such that* (23) *defines a well posed problem.*

We recall that the well-posedness of (23) requires at least that $n \neq 1$ in a neighborhood of ∂D and that k is outside a countable set without finite accumulation points.

3.2 Some useful properties for sampling methods

Let us define

$$\mathbb{Z}_M^{d-1} := \{j = (j_1, \cdots, j_{d-1}) \in \mathbb{Z}^{d-1}, [-\tfrac{M_\ell}{2}] + 1 \leq j_\ell \leq [\tfrac{M_\ell}{2}],\ \ell = 1, \cdots, d-1\}.$$

Most of our results are based on the assumption that

$$\Omega_M \setminus D \text{ is connected an } \partial\Omega_m \cap \overline{D} = \emptyset \text{ for all } m \in \mathbb{Z}_M^{d-1}.$$

The last assumption can be avoided with minor adaptations by changing the structure of the interior transmission problem. This assumption also justifies the use of N^+ or N^- and not both of them. We made the choice to adopt this assumption in order to avoid unnecessary additional technicalities.

A first step towards the justification of the sampling methods is the characterization of the closure of the range of $\mathcal{H}^\pm$.

Lemma 3.3. *The operator $\mathcal{H}^\pm$ is compact and injective. Let $H^\pm_{\mathrm{inc}}(D)$ be the closure of the range of $\mathcal{H}^\pm$ in $L^2(D)$. Then*

$$H^\pm_{\mathrm{inc}}(D) = H_{\mathrm{inc}}(D) := \{v \in L^2(D) : \ \ \Delta v + k^2 v = 0 \ \ in\ D\}. \tag{24}$$

Proof. We shall prove this lemma only for $\mathcal{H}^+$ since the proof for $\mathcal{H}^-$ is similar. Let $a = \{a(j)\}_{j\in\mathbb{Z}^{d-1}} \in \ell^2(\mathbb{Z}^{d-1})$ and assume that $\mathcal{H}^+ a = 0$ in D. Since,

$$\Delta(\mathcal{H}^+ a) + k^2(\mathcal{H}^+ a) = 0 \quad \text{ in } \ \mathbb{R}^3$$

then by the unique continuation principle, $\mathcal{H}^+ a = 0$ in $\mathbb{R}^3$. This implies that

$$0 = (\mathcal{H}^+ a)(\overline{x}, h) = -\frac{\mathrm{i}}{2} \sum_{j\in\mathbb{Z}^{d-1}} \frac{a(j)}{\overline{\beta_\#(j)}} e^{\mathrm{i}\alpha_\#(j)\cdot\overline{x}}$$

for all $\overline{x} \in \mathbb{R}^{d-1}$. This implies, using the inverse Fourier transform that $a_j = 0$ for all $j \in \mathbb{Z}^{d-1}$, which proves the injectivity of $\mathcal{H}^+$.
We now prove identity (24). We first obviously see that $H^+_{\mathrm{inc}}(D) \subset H_{\mathrm{inc}}(D)$. To prove the identity (24) it is then sufficient to prove that the adjoint $(\mathcal{H}^+)^*$ is injective on $H_{\mathrm{inc}}(D)$. Let $f \in H_{\mathrm{inc}}(D)$ and assume that $(\mathcal{H}^+)^*(f) = 0$. We set

$$u(x) := \int_D G_M(x,y)\, f(y)\, dy, \quad x \in \mathbb{R}^3, \tag{25}$$

where G_M is the $ML-$periodic Green function that has the expansion (12). From the regularity properties of volume potentials, we infer that $u \in H^2_{\#,\mathrm{loc}}(\Omega_M)$ and satisfies

$$\begin{cases} \Delta u + k^2 u = & -f \qquad \text{in } \ D, \\ \Delta u + k^2 u = & \ \ 0 \qquad \text{in } \ \Omega_M \setminus \overline{D}. \end{cases} \tag{26}$$

From expansion (12) and the definition of u in (25) we have that

$$\begin{aligned} \widehat{u}^+(j) &= \int_{[\![M_L^-, M_L^+]\!]} \int_D \frac{\mathrm{i}}{2[\![ML]\!]} \sum_{\ell\in\mathbb{Z}^{d-1}} \frac{1}{\beta_\#(\ell)} e^{\mathrm{i}\alpha_\#(\ell)(\overline{x}-\overline{y}) + \mathrm{i}\beta_\#(\ell)(h-y_d)} f(y)\, dy\, e^{-\mathrm{i}\alpha_\#(j)\cdot\overline{x}}\, \mathrm{d}\overline{x} \\ &= \int_D f(y) \frac{\mathrm{i}}{2\beta_\#(j)} e^{\mathrm{i}\alpha_\#(j) - \mathrm{i}\beta_\#(j)(x_d - h)} = \big((\mathcal{H}^+)^*(f)\big)(j), \end{aligned} \tag{27}$$

i.e., $(\mathcal{H}^+)^*(f) = \{\widehat{u}^+(j)\}_{j\in\mathbb{Z}^{d-1}}$, the Rayleigh sequence of u. Therefore, the assumption $(\mathcal{H}^+)^*(f) = 0$ implies that $\widehat{u}^+(j) = 0$ for all $j \in \mathbb{Z}^{d-1}$ and therefore $u = 0$ for all $x_d > h$. By the unique continuation principal and since $\Omega_M \setminus \overline{D}$ is connected, we have that $u = 0$ in $\Omega_M \setminus D$. The regularity $u \in H^2_{\#,\mathrm{loc}}(\Omega_M)$ then implies $u \in H^2_0(D)$. Taking the $L^2(D)$ scalar product of the first equation in (26) with f we obtain

$$\int_D \left(\Delta u + k^2 u\right) \overline{f}\, dx = \|f\|^2_{L^2(D)}\,.$$

Since $\Delta f + k^2 f = 0$ in D in the sense of distributions and since $u \in H^2_0(D)$, then

$$\int_D \left(\Delta u + k^2 u\right) \overline{f}\, dx = 0,$$

which proves that $f = 0$ and yields the injectivity of $(\mathcal{H}^+)^*$ on $H_{\mathrm{inc}}(D)$. □

Lemma 3.3 shows in particular that the closure of the range of $\mathcal{H}^\pm$ are identical and will be denoted in the sequel by $H_{\mathrm{inc}}(D)$. The following reciprocity lemma will also be useful.

Lemma 3.4. *Let $f_0, f_1 \in L^2(D)$ and let w_0 and $w_1 \in H^1_\#(\Omega^h_M)$ be the corresponding solutions satisfying* (10) *with $f = f_0$, $f = f_1$ respectively. Then*

$$\int_D (1-n) w_0 f_1 dx = \int_D (1-n) w_1 f_0 dx. \tag{28}$$

Proof. Taking $v = w_0$ and $v = w_1$ in the variational formulation (10) satisfied by w_1 and w_0 respectively then taking the difference yields

$$\int_{\Gamma^h_M} \frac{\partial w_0}{\partial x_2} w_1 - \frac{\partial w_1}{\partial x_2} w_0 \, ds(x) - \int_{\Gamma^{-h}_M} \frac{\partial w_0}{\partial x_2} w_1 - \frac{\partial w_1}{\partial x_2} w_0 \, ds(x)$$
$$= k^2 \int_D (1-n) f_0\, w_1 - (1-n) f_1\, w_0 \, dx. \tag{29}$$

Obviously $T^\pm$ are symmetric, since (using Parceval's theorem)

$$\pm \int_{\Gamma^{\pm h}_M} \frac{\partial w_0}{\partial x_2} w_1 ds(x) = \sum_{\ell \in \mathbb{Z}^{d-1}} \mathrm{i}\beta_\#(\ell) \widehat{w_0}^{\pm}(\ell) \widehat{w_1}^{\pm}(-\ell) \tag{30}$$

and $\beta_\#(\ell) = \beta_\#(-\ell)$ for all $\ell \in \mathbb{Z}^{d-1}$. Therefore,

$$\pm \int_{\Gamma^{\pm h}_M} \Big(\frac{\partial w_0}{\partial x_2} w_1 - \frac{\partial w_1}{\partial x_2} w_0 \Big)\, ds(x) = 0,$$

which proves the lemma. □

We now prove one of the main ingredients for the justification of the inversion methods discussed below. From now on, for $z \in \Omega^h_M$, we denote $\Phi(\cdot; z) := G_M(\cdot - z)$ which has the Rayleigh sequences $\widehat{\Phi}^\pm(\cdot; z) := \{\widehat{\Phi}^\pm(\ell; z)\}_{\ell \in \mathbb{Z}^{d-1}}$ with

$$\widehat{\Phi}^\pm(\ell; z) := \tfrac{\mathrm{i}}{2[\![ML]\!]\beta_\#(\ell)} e^{-\mathrm{i}(\alpha_\#(\ell)\overline{z} - \beta_\#(\ell)|z_d \mp h|)}.$$

Theorem 3.5. *Assume that Assumptions 2.1 and 3.2 hold. Then the operator* $\mathrm{G}^\pm : H_{\mathrm{inc}}(D) \to \ell^2(\mathbb{Z}^{d-1})$ *defined by (17) is injective with dense range. Moreover $\widehat{\Phi}^\pm(\cdot; z)$ belongs to $\mathcal{R}(\mathrm{G}^\pm)$ if and only if $z \in D$.*

Proof. We only give here the proof of theorem for the operator G^+ since the proof for the operator G^- is similar. We start by proving that $\mathrm{G}^+ : H_{\mathrm{inc}}(D) \to \ell^2(\mathbb{Z}^{d-1})$ is injective with dense range. Let $f \in H_{\mathrm{inc}}(D)$ and let $w \in H^1_\#(\Omega^h_M)$ be the associated scattered field via (10). As observed earlier, $w \in H^2_{\#,\mathrm{loc}}(\Omega_M)$. Assume that $\mathrm{G}^\pm(f) = 0$. Then $w = 0$ for $x_d > h$. By unique continuation principal we then deduce that

$$w = 0 \quad \text{in} \ \ \Omega_M \setminus D,$$

and therefore $w \in H^2_0(D)$. We now set, $u := w + f$, then the pair (u, f) is a solution to (23) with zero data. Assumption 3.2 then ensures that $f = 0$, which proves the injectivity of G^+. We now prove the denseness of the range of G^+. Let $g \in \overline{\mathcal{R}(\mathrm{G}^+)}^{\perp}$. Then

$$\big(\mathrm{G}^+(f), g\big)_{\ell^2(\mathbb{Z}^{d-1})} = 0,\ \forall f \in H_{\mathrm{inc}}(D).$$

Consider f of the form $f = \overline{\mathcal{H}^+(a)}$ for some $a \in \ell^2(\mathbb{Z}^{d-1})$. Since $\mathrm{G}^+ = (\mathcal{H}^+)^*\mathrm{T}$, we then have

$$\langle \mathrm{T}(\overline{\mathcal{H}^+(a)}), \mathcal{H}^+(g)\rangle_{L^2(D)} = 0,\ \forall a \in \ell^2(\mathbb{Z}^{d-1}). \tag{31}$$

Let $w(a)$ and $w(g)$ solution to (10) with respect to $\overline{\mathcal{H}^+(a)}$ and $\overline{\mathcal{H}^+(g)}$. From Lemma 3.4 we get

$$\begin{aligned}\Big(\mathrm{T}(\overline{\mathcal{H}^+(a)}), \mathcal{H}^+(g)\Big)_{L^2(D)} &= k^2 \int_D (n-1)(\overline{\mathcal{H}^+ a} + w(a))\, \overline{\mathcal{H}^+ g}\, dx \\ &= k^2 \int_D (n-1)(\overline{\mathcal{H}^+ g} + w(g))\, \overline{\mathcal{H}^+ g}\, dx.\end{aligned}$$

Therefore,

$$\Big(\mathrm{T}(\overline{\mathcal{H}^+(g)}), \mathcal{H}^+(a)\Big)_{L^2(D)} = \Big(\mathrm{T}(\overline{\mathcal{H}^+(a)}), \mathcal{H}^+(g)\Big)_{L^2(D)}, \quad \forall\, a \in \ell^2(\mathbb{Z}^{d-1}).$$

We deduce from (31) that

$$\Big(\mathrm{G}^+(\overline{\mathcal{H}^+(g)}), a\Big)_{\ell^2(\mathbb{Z}^{d-1})} = 0,\ \forall a \in \ell^2(\mathbb{Z}^{d-1}),$$

which implies $\mathrm{G}^+(\overline{\mathcal{H}^+(g)}) = 0$. The injectivity of G^+ ensures that $\mathcal{H}^+ g = 0$ in D and consequently $g = 0$ (by Lemma 3.3). This proves the denseness of the range of G^+.

We now prove the last part of the theorem. We first observe that $\widehat{\Phi}^+(\cdot; z)$ is the Rayleigh sequence of $\Phi(\cdot; z)$ satisfying $\Delta\Phi(\cdot; z) + k^2\Phi(\cdot; z) = -\delta_z$ in Ω_M and the Rayleigh radiation condition. Let $z \in D$. We consider $(u, v) \in L^2(D) \times L^2(D)$ as being the solution to (23) with

$$\varphi(x) = \Phi(x; z) \text{ and } \psi(x) = \partial\Phi(x; z)/\partial\nu(x) \quad \text{for } x \in \partial D. \tag{32}$$

We then define w by

$$\begin{aligned} w(x) = &\quad u(x) - v(x) \quad \text{in } D,\\ w(x) = &\qquad \Phi(x; z) \qquad \text{in } \Omega_M \setminus D.\end{aligned}$$

Due to (32), we have that $w \in H^2_{\#,\mathrm{loc}}(\Omega_M)$ and satisfies (10). Hence $\mathrm{G}^+ v = \widehat{\Phi}^+(\cdot; z)$.

Now let $z \in \Omega_M \setminus D$. Assume that there exists $f \in H_{\mathrm{inc}}(D)$ such that $\mathrm{G}^+ f = \widehat{\Phi}^+(\cdot; z)$. This implies that $w = \Phi(\cdot; z)$ in $\{x \in \Omega_M, \pm x_d \geq h\}$ where w is the solution to (10). By the unique continuation principle we deduce that $w = \Phi(\cdot; z)$ in $\Omega_M \setminus D$. This gives a contradiction since w is regular (locally H^2) in the neighborhood of z while $\Phi(\cdot; z)$ is not. $\square$

Lemma 3.6. *Let O be an open domain such that $\overline{O} \subset D$ and assume that there exists a real valued function $n_0 \in L^\infty(D)$ such that $(n_0(x) - 1) \geq \sigma > 0$, $x \in D$ (respectively $(1 - n_0(x)) \geq \sigma > 0$, $x \in D$) and $\mathrm{Re}\, n = n_0$ in $D \setminus O$. Then the operator $\mathrm{T} : L^2(D) \to L^2(D)$ defined by (20) satisfies*

$$\mathrm{Im}\ (\mathrm{T}\phi, \phi) \geq 0, \quad \forall \phi \in H_{\mathrm{inc}}(D) \tag{33}$$

and $\mathrm{Re}\,\mathrm{T} = \mathrm{T}_0 + \mathrm{T}_1$, where T_0 (respectively $-\mathrm{T}_0$) is self-adjoint and coercive and T_1 is compact on $H_{\mathrm{inc}}(D)$. Moreover, assume in addition that Assumption (3.2) holds or $(n-1)^{-1} \in L^\infty(D)$. Then T is injective on $H_{\mathrm{inc}}(D)$

Proof. Let $\phi \in L^2(D)$ and w_ϕ be solution to (10) associated with $f = \phi$. By definition of the operator T we have

$$(\mathrm{T}\phi, \phi)_{L^2(D)} = k^2 \int_D (n-1)(\phi + w_\phi)\overline{\phi} = k^2 \int_D (n-1)|\phi + w_\phi|^2 - k^2 \int_D (n-1)(w_\phi + \phi)\overline{w_\phi} \tag{34}$$

where, using the variational formulation (10),

$$\begin{aligned} - k^2 \int_D (n-1)(w_\phi + \phi)\overline{w_\phi} &= \int_D (\Delta w_\phi + w_\phi)\overline{w_\phi} \\ &= \langle T^+ w_\phi, w_\phi\rangle + \langle T^- w_\phi, w_\phi\rangle - \int_D |\nabla w_\phi|^2 - k^2|w_\phi|^2 \,\mathrm{d}x \end{aligned} \tag{35}$$

Therefore,

$$\begin{aligned} \langle \mathrm{T}\phi, \phi\rangle = k^2 \int_D (n-1)|\phi + w_\phi|^2 \\ - \int_D |\nabla w_\phi|^2 - k^2|w_\phi|^2 + \langle T^+ w_\phi, w_\phi\rangle + \langle T^- w_\phi, w_\phi\rangle \end{aligned} \tag{36}$$

Thanks to the non-negative sign of the imaginary part of $T^\pm$ and the assumption $\mathrm{Im}\,(n) \geq 0$ we deduce that

$$\mathrm{Im}\,\langle \mathrm{T}\phi, \phi\rangle = k^2 \int_D \mathrm{Im}\,(n)|w_\phi + \phi|^2 + \mathrm{Im}\,\langle T^+ w_\phi, w_\phi\rangle + \mathrm{Im}\,\langle T^- w_\phi, w_\phi\rangle \geq 0. \tag{37}$$

Define $\mathrm{T}_0 : L^2(D) \to L^2(D)$ by

$$\mathrm{T}_0\phi := k^2(n_0 - 1)\phi. \tag{38}$$

Clearly, T_0 (respectively $-\mathrm{T}_0$) is self-adjoint and coercive. Moreover, $\mathrm{T}_1 = \mathrm{T} - \mathrm{T}_0$ satisfies, $\mathrm{T}_1\phi = k^2(1-n)w_\phi + k^2(n_0 - \mathrm{Re}\,n)\phi$. The application $\phi \mapsto w_\phi$ is continuous from $L^2(D)$ into $H^2(D)$ and since for $\phi \in H_{\mathrm{inc}}(D)$, $\Delta\phi + k^2\phi = 0$ in D, the application that $\phi \mapsto \phi|_O$ is continuous from $H_{\mathrm{inc}}(D)$ into $H^2(O)$. Therefore, the operator $\mathrm{T}_1 : H_{\mathrm{inc}}(D) \to L^2(D)$ is compact thanks to the Rellich compact embedding theorem.

Assume that $\phi \in H_{\mathrm{inc}}(D)$ and $T\phi = k^2(n-1)(\phi + w_\phi) = 0$. This implies that $w_\phi = 0$ by uniqueness of solutions to problem (10) with $n = 1$.
If we assume that in addition $(n-1)^{-1} \in L^\infty(D)$. Then $T\phi = 0$ also implies $\phi + w_\phi = 0$ in D and therefore $\phi = 0$.
The injectivity of T is remains true if Assumption 3.2 holds. With $\phi \in H_{\mathrm{inc}}(D)$ verifying $\Delta\phi + k^2\phi = 0$ in D we get that $u := \phi + w_\phi$ and $v := \phi$ are such that $u - v \in H_0^2(D)$ and satisfy the interior transmission problem (23) with $\varphi = \psi = 0$. We then deduce that $u = v = 0$. This concludes the proof of the injectivity of the operator T. □

Lemma 3.7. *Assume that Assumptions* (2.1) *and* (3.2) *hold. Then the operators* $\mathrm{N}^\pm$ *are injective with dense ranges.*

Proof. The injectivity and the denseness of the ranges of $\mathrm{N}^\pm$ directly follow from the same properties satisfied by $\mathcal{H}^\pm$ (Lemma 3.3) and $\mathrm{G}^\pm$ (Theorem 3.5). □

4 Application to Sampling methods

We shall provide here the theoretical justifications of three sampling methods : the Linear Sampling Method, the Factorization Method and the Generalized Linear Sampling Method, to reconstruct the domain D from one of the near field operators $\mathrm{N}^\pm$. These justifications are mainly a direct application of the results of the previous section and the abstract theoretical framework of these methods that is recalled in the appendix. This section is preparatory to the next section where we propose a new algorithm capable of reconstructing directly the domain ω from $\mathrm{N}^\pm$.

4.1 The Linear Sampling Method (LSM)

We give here the classical justification for the use of so-called Linear Sampling Method (LSM). This justification is a consequence of Theorem 3.5 and the Lemma 3.3. Since the operator $\mathcal{H}^{\pm}$ is compact, the characterization of D in terms of the range of $\mathrm{G}^{\pm}$ in Theorem 3.5 does not imply a similar characterization in terms of the range of $\mathrm{N}^{\pm}$. However one can deduce the following.

Theorem 4.1. *Assume that Assumptions* (2.1) *and* (3.2) *hold. Then:*

- *If $z \in D$ then there exists a sequence $a_\alpha^{\pm}(z) \in \ell^2(\mathbb{Z}^{d-1})$ such that* $\lim_{\alpha\to 0} \|\mathrm{N}^{\pm}(a_\alpha^{\pm}(z)) - \widehat{\Phi}^{\pm}(\cdot;z)\|_{\ell^2(\mathbb{Z}^{d-1})} = 0$ *and* $\lim_{\alpha\to 0} \|\mathcal{H}^{\pm} a_\alpha^{\pm}(z)\|_{L^2(D)} < \infty$.
- *If $z \notin D$ then for all $a_\alpha^{\pm}(z) \in \ell^2(\mathbb{Z}^{d-1})$ such that* $\lim_{\alpha\to 0} \|\mathrm{N}^{\pm}(a_\alpha^{\pm}(z)) - \widehat{\Phi}^{\pm}(\cdot;z)\|_{\ell^2(\mathbb{Z}^{d-1})} \to 0$, $\lim_{\alpha\to 0} \|\mathcal{H}^{\pm} a_\alpha^{\pm}(z)\|_{L^2(D)} = \infty$.

Proof. The proof is classical but we give it here for the reader's convenience.

If $z \in D$ then let $f \in H_{\mathrm{inc}}(D)$ be such that $\mathrm{G}^{\pm} f = \widehat{\Phi}^{\pm}(\cdot;z)$ which exists by Theorem 3.5. From Lemma 3.3 there exists a sequence $a_z^{\alpha} \in \ell^2(\mathbb{Z}^{d-1})$ such that $\mathcal{H}^{\pm} a_z^{\alpha} \to f$ as $\alpha \to 0$, and the first statement follows from the fact that $\mathrm{N}^{\pm} = \mathrm{G}^{\pm}\, \mathcal{H}^{\pm}$.

Let $z \notin D$ and $a_\alpha^{\pm}(z) \in \ell^2(\mathbb{Z}^{d-1})$ be such that $\lim_{\alpha\to 0} \|\mathrm{N}^{\pm} a_\alpha^{\pm}(z) - \widehat{\Phi}^{\pm}(\cdot;z)\|_{\ell^2(\mathbb{Z}^{d-1})} \to 0$. Assume that $\|\mathcal{H}^{\pm} a_\alpha^{\pm}(z)\|_{L^2(D)}$ is bounded as $\alpha \to 0$. We can assume that $\mathcal{H}^{\pm} a_\alpha^{\pm}(z)$ weakly converges to some $f \in H_{\mathrm{inc}}(D)$. Since $\mathrm{G}^{\pm}\, \mathcal{H}^{\pm} = \mathrm{N}^{\pm}$ then we get as $\alpha \to 0$, $\mathrm{G}^{\pm} f = \widehat{\Phi}^{\pm}(\cdot;z)$ which contradicts the last part of Theorem 3.5. $\square$

This theorem does not indicate how to construct the sequence $a_\alpha^{\pm}(z)$ when $z \in D$. In practice one relies on the use of Tikhonov regularization and considers $\tilde{a}_\alpha^{\pm}(z) \in \ell^2(\mathbb{Z}^{d-1})$ satisfying

$$(\alpha + (\mathrm{N}^{\pm})^* \mathrm{N}^{\pm})\; \tilde{a}_\alpha^{\pm}(z) = (\mathrm{N}^{\pm})^* \left(\widehat{\Phi}^{\pm}(\cdot;z)\right). \tag{39}$$

Since $\mathrm{N}^{\pm}$ has dense range, $\lim_{\alpha\to 0} \|\mathrm{N}^{\pm} \tilde{a}_\alpha^{\pm}(z) - \widehat{\Phi}^{\pm}(\cdot;z)\|_{\ell^2(\mathbb{Z}^{d-1})} = 0$. However, one cannot guarantee in general that $\lim_{\alpha\to 0} \|\mathcal{H}^{\pm} \tilde{a}_\alpha^{\pm}(z)\|_{L^2(D)} < \infty$ if $z \in D$. In addition, one cannot compute $\|\mathcal{H}^{\pm} a_\alpha^{\pm}(z)\|_{L^2(D)}$ since D is not known. In practice one uses $z \to \|a_\alpha^{\pm}(z)\|_{\ell^2(\mathbb{Z}^{d-1})}$ as an indicator function for D. A possible method to fix the Tikhonov regularization parameter α in (39) is to use the Morozov discrepancy principle. Assume that $\mathrm{N}^{\pm,\delta}$ is the noisy operator corresponding to noisy measurements, i.e

$$\|\mathrm{N}^{\pm,\delta} - \mathrm{N}^{\pm}\| \le \delta.$$

Then for each sampling point z, the parameter α is chosen such that

$$\|\mathrm{N}^{\pm,\delta} a_\alpha^{\pm}(z) - \widehat{\Phi}^{\pm}(\cdot;z)\|_{\ell^2(\mathbb{Z}^{d-1})} = \delta \|a_\alpha^{\pm}(z)\|_{\ell^2(\mathbb{Z}^{d-1})}.$$

This leads to a nonlinear equation that determines α in terms of the noise level δ [16].

4.2 The Factorization Method

We here proceed with the justification of the Factorization method that has been introduced in [23] and that was applied in a number of papers to various configurations [2,3,7,14,18,19,21,24–29,33]. Our setting is similar to the case of guided waves that has been treated in [8] or the case of periodic media in [30] where the half space problem was considered. We include the analysis of this method here since we shall prove the counterpart for a single Floquet-Bloch mode later. Let us define the operator

$$\mathrm{N}_\sharp^{\pm} := |\mathrm{Re}\,(\mathrm{N}^{\pm})| + |\mathrm{Im}\,(\mathrm{N}^{\pm})| \tag{40}$$

where $\mathrm{Re}\,(\mathrm{N}^{\pm}) := \frac{1}{2}\,(\mathrm{N}^{\pm} + (\mathrm{N}^{\pm})^*)$, $\mathrm{Im}\,(\mathrm{N}^{\pm}) := \frac{1}{2\mathrm{i}}\,(\mathrm{N}^{\pm} - (\mathrm{N}^{\pm})^*)$. Then we have the following theorem:

Theorem 4.2. *Under the hypothesis of Lemma 3.6, the following factorization holds:*

$$\mathrm{N}_\sharp^\pm = (\mathcal{H}^\pm)^* \, \mathrm{T}_\sharp^\pm \, \mathcal{H}^\pm, \tag{41}$$

where $\mathrm{T}_\sharp^\pm : L^2(D) \to L^2(D)$ *is self-adjoint and coercive on* $H_{\mathrm{inc}}(D)$*. Moreover,* $z \in D$ *if and only if* $\widehat{\Phi}^\pm(\cdot; z) \in \mathcal{R}\left((\mathrm{N}_\sharp^\pm)^{1/2}\right)$.

Proof. The proof of this theorem is a direct application of the abstract framework given in Theorem A.2 using the results of Lemma 3.6 and Lemma 4.3 below. □

Lemma 4.3. *For* $z \in \Omega_M$*,* $z \in D$ *if and only if* $\widehat{\Phi}^\pm(\cdot; z)$ *belongs to the range of* $(\mathcal{H}^\pm)^*$.

Proof. For $z \in D$ choose a cut-off function $\rho \in C^\infty(\Omega_M)$ which vanishes near z and equals one in $\Omega_M \backslash D$. We define $v(x) := \rho(x) G_{M,z}$. Then the Rayleigh sequence of $v(x)$ are $\widehat{\Phi}^\pm(\cdot; z)$. We observe that $f := -(\Delta v + k^2 v)$ has compact support in D and $f \in L^2(D)$. Since v satisfies the Rayleigh radiation condition, then

$$v(x) = \int_D G_M(x - y) f(y) dy. \tag{42}$$

We hence have from expansion (12) and the fact the Rayleigh sequences of v and $\Phi(\cdot; z)$ are identical that

$$\widehat{\Phi}^\pm(\cdot; z) = (\mathcal{H}^\pm)^* f.$$

We now assume that $z \notin D$ (without loss of generality we can assume that $D \cup \{z\} \subset \Omega_M^h$) and $\widehat{\Phi}^\pm(\cdot; z) = (\mathcal{H}^\pm)^* f$ for some $f \in L^2(D)$. We also consider v which is defined by (42). Since $\Phi(\cdot; z)$ and v satisfies the Rayleigh radiation condition then $\Phi(\cdot; z) = v$ in domain $\pm x_d > h$. By unique continuation principal we deduce that $\Phi(\cdot; z) = v$ in the exterior of $D \cup \{z\}$. This gives a contradiction since v is smooth near z but $\Phi(\cdot; z)$ is singular at z. □

We can also reformulate the second part of Theorem 4.2 by using Picard's criterion: $z \in D$ if and only if the series

$$\sum_{m=1}^{\infty} \frac{\left|(\widehat{\Phi}^\pm(\cdot; z), \Psi_m^\pm)_{\ell^2(\mathbb{Z}^{d-1})}\right|^2}{\lambda_m^\pm} \tag{43}$$

converges, where $\{\lambda_m^\pm, \Psi_m^\pm\}$ is the eigensystem of the self-adjoint and positive defined $\mathrm{N}_\sharp^\pm$. This criterion can also be used in the numerical implementation of the factorization method with a suitable choice of the cut-off parameter with respect to the noise level. One can also rely on the use of Tikhonov regularization as explained above for the linear sampling method.

4.3 The Generalized Linear Sampling Method (GLSM)

This section is dedicated to the third family of sampling methods that has been introduced in the literature [4,6,11] and that somehow combines the benefits from the two previously presented sampling methods. The GLSM constructs a nearby solution as predicted by the LSM theorem by considering minimizing sequences of a cost functional with data fidelity the LSM residual and a penalty term the norm of the Herglotz function. The latter is constructed exploiting the factorization method. The first advantage of the GLSM is indeed to have a more convincing theoretical justification than LSM. Compared to the factorization method, the GLSM keep the link with the so-called interior transmission problem as for LSM which will be exploited later for the design of the new imaging functional capable of directly identifying a defect in a periodic background. We here restrict ourselves to the simplest version of GLSM that exploits "symmetric factorizations" of the data operator. For the treatment of other type of factorizations we refer the reader to [4].

We first present the noise free version of GLSM. We denote by $(\cdot,\cdot)$ the $\ell^2(\mathbb{Z}^{d-1})$ scalar product and by $\|.\|$ the associated norm. Let $\alpha > 0$ be a given parameter and $\phi \in \ell^2(\mathbb{Z}^{d-1})$. We introduce functional $J_\alpha(\phi;\cdot) : \ell^2(\mathbb{Z}^{d-1}) \to \mathbb{R}$ where

$$J^\pm_\alpha(\phi; a) := \alpha(\mathrm{N}^\pm_\sharp a, a) + \left\|\mathrm{N}^\pm a - \phi\right\|^2, \quad \forall a \in \ell^2(\mathbb{Z}^{d-1}), \tag{44}$$

and define

$$j_\alpha(\phi) := \inf_{a\in\ell^2(\mathbb{Z}^{d-1})} J^\pm_\alpha(\phi; a). \tag{45}$$

Let $c(\alpha) > 0$ be such that $c(\alpha)/\alpha \to 0$ as $\alpha \to 0$.

Theorem 4.4. *Assume that Assumptions 2.1, 3.2 and the hypothesis of Lemma 3.6 hold. Let $z \in \Omega_M$ and define $a^\pm_{\alpha,G}(z) \in \ell^2(\mathbb{Z}^{d-1})$ such that*

$$J^\pm_\alpha(\widehat{\Phi}^\pm(\cdot; z); a^\pm_{\alpha,G}(z)) \leq j_\alpha(\widehat{\Phi}^\pm(\cdot; z)) + c(\alpha). \tag{46}$$

Then $z \in D$ if and only if $\lim_{\alpha\to0}(\mathrm{N}^\pm_\sharp a^\pm_{\alpha,G}(z), a^\pm_{\alpha,G}(z)) < \infty$. Moreover, if $z \in D$ then $\mathcal{H}^\pm a^\pm_{\alpha,G}(z) \to v(\cdot; z)$ in $L^2(D)$ where $(u(\cdot; z), v(\cdot; z)) \in L^2(D) \times L^2(D)$ is the solution of problem (23) *with $\varphi = \Phi(\cdot; z)$ and $\psi = \partial\Phi(\cdot; z)/\partial\nu$ on ∂D.*

Proof. The proof of this theorem is a direct application of the abstract framework given in Theorem A.4 in combination with Theorem A.2 □

For the case with noise in the operators, one has to change the functional $J^\pm_\alpha$. More precisely, consider the noisy operators $\mathrm{N}^{\pm,\delta} : \ell^2(\mathbb{Z}^{d-1}) \to \ell^2(\mathbb{Z}^{d-1})$ and ${\mathrm{N}^\pm_\sharp}^\delta : \ell^2(\mathbb{Z}^{d-1}) \to \ell^2(\mathbb{Z}^{d-1})$ such that

$$\|\mathrm{N}^{\pm,\delta} - \mathrm{N}^\pm\| \leq \delta\|\mathrm{N}^{\pm,\delta}\| \quad \text{and} \quad \|{\mathrm{N}^\pm_\sharp}^\delta - \mathrm{N}^\pm_\sharp\| \leq \delta\|{\mathrm{N}^\pm_\sharp}^\delta\|, \tag{47}$$

for some $\delta > 0$ and assume that the operators $\mathrm{N}^{\pm,\delta}$ and ${\mathrm{N}^\pm_\sharp}^\delta$ are compact. We then consider for $\alpha > 0$ and $\phi > 0$ the functional $J^\delta_\alpha(\phi;\cdot) : \ell^2(\mathbb{Z}^{d-1}) \to \mathbb{R}$ defined by

$$J^{\pm,\delta}_\alpha(\phi; a) := \alpha\left(({\mathrm{N}^\pm_\sharp}^\delta a, a) + \delta\alpha^{-\eta}\|{\mathrm{N}^\pm_\sharp}^\delta\|\,\|a\|^2\right) + \left\|\mathrm{N}^{\pm,\delta} a - \phi\right\|^2, \quad \forall\, a \in \ell^2(\mathbb{Z}^{d-1}), \tag{48}$$

with $0 < \eta < 1$ a fixed parameter. Then we have the following result:

Theorem 4.5. *Assume that Assumptions 2.1, 3.2 and the assumptions of Lemma 3.6 hold. For $z \in \Omega_M$ denote by $a^\pm_{\alpha,\delta}(z)$ the minimizer of $J^{\pm,\delta}_\alpha(\widehat{\Phi}^\pm(\cdot; z);\cdot)$ over $\ell^2(\mathbb{Z}^{d-1})$. Then,*

$$z \in D \text{ if and only if } \lim_{\alpha\to0}\limsup_{\delta\to0}\left(\left|({\mathrm{N}^\pm_\sharp}^\delta a^\pm_{\alpha,\delta}(z), a^\pm_{\alpha,\delta}(z))\right| + \delta\alpha^{-\eta}\|{\mathrm{N}^\pm_\sharp}^\delta\|\left\|a^\pm_{\alpha,\delta}(z)\right\|^2\right) < \infty.$$

Moreover, if $z \in D$ then $\lim_{\alpha\to0}\limsup_{\delta\to0}\|\mathcal{H}^\pm a^\pm_{\alpha,\delta}(z) - v(\cdot; z)\|_{L^2(D)} = 0$ where $(u(\cdot; z), v(\cdot; z)) \in L^2(D) \times L^2(D)$ is the solution to problem (23) *with $\varphi = \Phi(\cdot; z)$ and $\psi = \partial\Phi(\cdot; z)/\partial\nu$ on ∂D.*

Proof. This is a direct consequence of Theorem A.6. □

We remark that (for numerics) the minimizer $a^\pm_{\alpha,\delta}(z)$ in Theorem 4.5 can be computed as the solution of

$$(\alpha{\mathrm{N}^\pm_\sharp}^\delta + \alpha^{1-\eta}\delta\|{\mathrm{N}^\pm_\sharp}^\delta\| I + (\mathrm{N}^{\pm,\delta})^*\mathrm{N}^{\pm,\delta})a^\pm_{\alpha,\delta}(z) = (\mathrm{N}^{\pm,\delta})^*\widehat{\Phi}^\pm(\cdot; z). \tag{49}$$

Unfortunately, there is no a posteriori rule for the choice of the parameter α as for the LSM method. In practice, one can follow the empirical rule proposed in [6] by taking

$$\alpha \equiv \alpha(\delta)/(\|{\mathrm{N}^\pm_\sharp}^\delta\|(1 + \delta)) \tag{50}$$

where $\alpha(\delta)$ is the LSM regularization parameter determined by the Morozov principle as indicated in Section 4.1. The parameter η has little incidence on the numerics and can be set to 0. Therefore an approximate indicator function of the domain D is given by the function

$$z \mapsto \frac{1}{\left((\mathrm{N}_\sharp^{\pm\,\delta} a^\pm_{\alpha,\delta}(z), a^\pm_{\alpha,\delta}(z)) + \delta \|\mathrm{N}_\sharp^{\pm\,\delta}\| \left\| a^\pm_{\alpha,\delta}(z)\right\|^2\right)}. \tag{51}$$

4.4 Reconstruction of the periodic domain D_p from $\mathrm{N}^\pm$

We here explain how one can reconstruct the periodic background D_p from $\mathrm{N}^\pm$ without knowing the refractive indices n and n_p using the same sampling operators as above. To reconstruct D_p we do not need the local perturbation ω to be located in only one period but we need to assume that for all $z \in \omega$ there exists $m \in \mathbb{Z}^{d-1}$ such that $z + m\mathbf{e} \in \Omega_M \setminus D$. Here and in the following we denote by $\mathbf{e}$ one of the vectors $L_i \mathbf{e}_i$, $i = 1, \dots, d-1$ where $\mathbf{e}_1, \dots, \mathbf{e}_d$ is the canonical basis of $\mathbb{R}^d$.

We shall exploit the decomposition of the $ML-$periodic fundamental solution into L quasi-periodic fundamental solutions.

Definition 4.6. *A function u is called quasi-periodic with parameter $\xi = (\xi_1, \cdots, \xi_{d-1})$ and period $L = (L_1, \cdots, L_{d-1})$, with respect to the first $d-1$ variables (briefly denoted as $\xi-$quasi-periodic with period L) if:*

$$u(\overline{x} + (jL), x_d) = e^{i\xi\cdot(jL)} u(\overline{x}, x_d), \quad \forall j \in \mathbb{Z}^{d-1}.$$

We remark from using discrete Floquet-Bloch transform that an $ML-$periodic function can be decomposed into the sum of M quasi-periodic functions with period L and quasi-periodicity parameters $\alpha_q = 2\pi q/(ML)$, $q \in \mathbb{Z}^{d-1}_M$ as:

$$u = \frac{1}{[\![M]\!]} \sum_{q \in \mathbb{Z}^{d-1}_M} u_q \tag{52}$$

where u_q is α_q-quasi-periodic with period L. More precisely, for $u \in L^2(\Omega^h_M)$ that is decomposed as

$$u(\overline{x}, x_d) = \sum_{j \in \mathbb{Z}^{d-1}} \tilde{u}(j, x_d) e^{\mathrm{i}\alpha_\#(j)\overline{x}}$$

where

$$\tilde{u}(j, x_d) := \frac{1}{[\![ML]\!]} \int_{[\![M_L^-, M_L^+]\!]} u(\overline{x}, x_d) e^{-\mathrm{i}\alpha_\#(j)\overline{x}} d\overline{x},$$

we define u_q as:

$$u_q(\overline{x}, x_d) := [\![M]\!] \sum_{j \in \mathbb{Z}^{d-1}} \tilde{u}(q + Mj, x_d) e^{\mathrm{i}\alpha_\#(q+Mj)\overline{x}}.$$

Writing the decomposition

$$\Phi(\cdot; z) := \frac{1}{[\![M]\!]} \sum_{q \in \mathbb{Z}^{d-1}_M} \Phi_q(\cdot; z),$$

the functions $\Phi_q(\cdot; z)$ satisfies $\Delta\Phi_q(\cdot; z) + k^2 \Phi_q(\cdot; z) = -\delta_z$ in Ω_0 and is α_q quasi-periodic with period L. Moreover,

$$\Phi_q(\cdot; z + \mathbf{e}) = e^{-i\alpha_q \cdot \mathbf{e}} \Phi_q(\cdot; z), \quad z \in \mathbb{R}^d.$$

The Rayleigh coefficients $\widehat{\Phi}^\pm_q(\cdot; z)$ of $\Phi_q(\cdot; z)$ are given by

$$\widehat{\Phi}^\pm_q(j; z) = \begin{cases} \frac{\mathrm{i}}{2[\![L]\!]\beta_\#(q+M\ell)} e^{-\mathrm{i}(\alpha_\#(q+M\ell)\overline{z} - \beta_\#(q+M\ell)|z_d \mp h|)} & \text{if} \quad j = q + M\ell,\ \ell \in \mathbb{Z}^{d-1}, \\ 0 & \text{if} \quad j \neq q + M\ell,\ \forall \ell \in \mathbb{Z}^{d-1}. \end{cases}$$

Lemma 4.7. *Assume that Assumptions* (2.1) *and* (3.2) *hold. Then* $\widehat{\Phi}_q^{\pm}(\cdot;z)$ *belongs to* $\mathcal{R}(\mathrm{G}^{\pm})$ *if and only if* $z \in D_p$.

Proof. Consider first the case $z \in D_p$ and let $(u_q(\cdot;z), v_q(\cdot;z)) \in L^2(D) \times L^2(D)$ be the solution to the interior transmission problem (23) with

$$\varphi(x) = \Phi_q(x;z) \text{ and } \psi(x) = \partial\Phi_q(x;z)/\partial\nu(x) \quad \text{for } x \in \partial D. \tag{53}$$

We then define w_q by:

$$w_q(x) = \begin{cases} u_q(x;z) - v_q(x;z) & \text{in } D, \\ \Phi_q(x;z) & \text{in } \Omega_M \setminus D. \end{cases}$$

Due to (53), $w_q \in H^2_{\#,\mathrm{loc}}(\Omega_M)$ and since $\Phi_q(\cdot;z)$ satisfies $\Delta\Phi_q(\cdot;z) + k^2\Phi_q(\cdot;z) = -\delta_z$ in $\Omega_M \setminus D_p$ and the Rayleigh radiation condition (3), then w_q satisfies (10). Moreover, we get that $\widehat{\Phi}_q^{\pm}(\cdot;z)$ are the Rayleigh coefficients of w_q and consequently $\mathrm{G}^{\pm}v_q(x;z) = \widehat{\Phi}_q^{\pm}(\cdot;z)$.
Consider now the case $z \in \Omega_M \setminus D_p$. Assume that there exists $f \in H_{\mathrm{inc}}(D)$ such that $\mathrm{G}^{\pm}f = \widehat{\Phi}_q^{\pm}(\cdot;z)$. This implies that $w_q = \Phi_q(\cdot;z)$ in $\{x \in \Omega_M, \pm x_d \geq h\}$ where w_q is the solution to (10). By the unique continuation principal we deduce that $w_q = \Phi_q(\cdot;z)$ in $\Omega_M \setminus D$. Since there exists $m \in \mathbb{Z}^{d-1}$ such that $z + m\mathbf{e} \in \Omega_M \setminus D$ then $\Phi_q(\cdot;z)$ is not locally H^2 in $\Omega_M \setminus D$ (since $\Phi_q(\cdot;z)$ is singular at all points $z + m\mathbf{e}$). This contradicts the fact that w is locally H^2 in $\Omega_M \setminus D$. □

Lemma 4.8. *For* $z \in \Omega_M$, $z \in D_p$ *if and only if* $\widehat{\Phi}_q^{\pm}(\cdot;z)$ *belongs to the range of* $(\mathcal{H}^{\pm})^*$.

Proof. For $z \in D_p$, we choose a cut-off $L-$periodic function $\rho \in C^\infty(\Omega_M)$ that vanishes in a neighborhood of z and equals one in $\Omega_M \setminus D_p$. We define $v(x) := \rho(x)\Phi_q(\cdot;z)$. Then $v(x)$ is α_q-quasi-periodic and the Rayleigh sequence of $v(x)$ is equal to $\widehat{\Phi}_q^{\pm}(\cdot;z)$. We observe that $f := -(\Delta v + k^2 v)$ has compact support in D_p and $f \in L^2(D)$. Since v satisfies the Rayleigh radiation condition and is $ML-$periodic, then

$$v(x) = \int_{D_p} \Phi_q(x-y)f(y)dy = [\![M]\!] \int_{D_p} \Phi(x-y)f(y)dy = [\![M]\!] \int_{D} \Phi(x-y)f(y)dy. \tag{54}$$

For the first equality we used the fact f is α_q-quasi-periodic function and therefore

$$v(x) = \int_{D_p} \Phi_{q'}(x-y)f(y)dy = 0, \quad \forall q' \in \mathbb{Z}_M^{d-1},\ q' \neq q \tag{55}$$

while for the second equality we simply used $f = 0$ in $D \setminus D_p$. From the expansion

$$\Phi_q(x) = \frac{\mathrm{i}}{2[\![L]\!]} \sum_{\ell \in \mathbb{Z}^{d-1}} \frac{1}{\beta_\#(q+M\ell)} e^{\mathrm{i}\alpha_\#(q+M\ell)\overline{x} + \mathrm{i}\beta_\#(q+M\ell)|x_d|} \tag{56}$$

and the fact that the Rayleigh sequences of v and $\Phi_q(\cdot;z)$ are identical we obtain that

$$\widehat{\Phi}_q^{\pm}(\cdot;z) = (\mathcal{H}^{\pm})^*(Mf).$$

We now assume that $z \notin D_p$ (without loss of generality we can assume that $D_p \cup \{z\} \subset \Omega_M^h$) and $\widehat{\Phi}_q^{\pm}(\cdot;z) = (\mathcal{H}^{\pm})^* f$ for some $f \in L^2(D)$. We also consider v which is defined by (54). Since $\Phi_q(\cdot;z)$ and v satisfies the Rayleigh radiation condition then $\Phi_q(\cdot;z) = v$ in the domain $\pm x_d > h$. By the unique continuation principle, we deduce that $\Phi_q(\cdot;z) = v$ in the exterior of $D \cup \{z + mL\}$, $m \in \mathbb{Z}_M^{d-1}$. This gives a contradiction since v is a smooth function in the neighborhood of the points $\{z + mL\}$, $m \in \mathbb{Z}_M^{d-1}$ but $\Phi_q(\cdot;z)$ is not. □

The previous two results allow us to conclude that one can reconstruct the domain D_p using the sampling methods introduced above by replacing the sampling function $\widehat{\Phi}^{\pm}(\cdot;z)$ with $\widehat{\Phi}_q^{\pm}(\cdot;z)$.

Theorem 4.9. *Let* q *be a fixed parameter in* $\mathbb{Z}_M^{d-1}$. *Assume that Assumptions 2.1, 3.2 and the assumptions of Lemma 3.6 hold. Then Theorems 4.2, 4.4 and 4.5 hold true if* D *is replaced by* D_p *and* $\widehat{\Phi}^{\pm}(\cdot;z)$ *is replaced by* $\widehat{\Phi}_q^{\pm}(\cdot;z)$.

5 Sampling methods for a single Floquet-Bloch mode

The first step toward the construction of our indicator function of ω without the need for a measurement operator for the background is to construct a sampling operator that would roughly speaking plays the role of the background sampling operator. A natural candidate for this operator is the one obtained from the operator $\mathcal{H}$ by restricting to single Floquet-Bloch modes. This choice comes from the fact that in the periodic case, i.e. $n = n_p$, this type of operators have been used to reconstruct the background medium (see [31]). As we can shall see later, for the case of perturbed periodic media, this operator reconstruct the background media and the perturbation with periodic copies of period L.

5.1 Near field operator for a fixed Floquet-Bloch mode

We shall present in this section how to define the sampling operator for one fixed Floquet-Bloch mode. Let $a \in \ell^2(\mathbb{Z}^{d-1})$, we define for $q \in \mathbb{Z}_M^{d-1}$, the element $a_q \in \ell^2(\mathbb{Z}^{d-1})$ by

$$a_q(j) := a(q + jM).$$

We then define the operator $\mathrm{I}_q : \ell^2(\mathbb{Z}^{d-1}) \to \ell^2(\mathbb{Z}^{d-1})$, which transforms $a \in \ell^2(\mathbb{Z}^{d-1})$ to $\tilde{a} \in \ell^2(\mathbb{Z}^{d-1})$ such that

$$\tilde{a}_q = a \quad \text{and} \quad \tilde{a}_{q'} = 0 \ \text{ if } \ q \neq q'.$$

We remark that $\mathrm{I}_q^*(a) = a_q$, where $\mathrm{I}_q^* : \ell^2(\mathbb{Z}^{d-1}) \to \ell^2(\mathbb{Z}^{d-1})$ is the dual of the operator I_q. The single Floquet-Bloch mode Herglotz operator $\mathcal{H}_q^\pm : \ell^2(\mathbb{Z}^{d-1}) \to L^2(D)$ is defined by

$$\mathcal{H}_q^\pm a := \mathcal{H}^\pm \mathrm{I}_q a = \sum_j a(j) u^{i,\pm}(\cdot; q + jM)|_D \tag{57}$$

and the single Floquet-Bloch mode sampling operator $\mathrm{N}_q^\pm : \ell^2(\mathbb{Z}^{d-1}) \to \ell^2(\mathbb{Z}^{d-1})$ is defined by

$$\mathrm{N}_q^\pm \, a = \mathrm{I}_q^* \, \mathrm{N}^\pm \, \mathrm{I}_q \, a. \tag{58}$$

We remark that $\mathcal{H}_q^\pm a$ is an α_q-quasi-periodic function with period L. The sequence $\mathrm{N}_q^\pm \, a$ corresponds with the Fourier coefficients of the α_q-quasi-periodic component of the scattered field in the decomposition (52). This operator is then somehow associated with α_q-quasi-periodicity.
One immediately see from the factorization $\mathrm{N}^\pm = (\mathcal{H}^\pm)^* \, \mathrm{T} \, \mathcal{H}^\pm$ that the following factorization holds.

$$\mathrm{N}_q^\pm = (\mathcal{H}_q^\pm)^* \, \mathrm{T} \, \mathcal{H}_q^\pm. \tag{59}$$

For later use we also define the operator $\mathrm{G}_q^\pm : \overline{\mathcal{R}(\mathcal{H}_q^\pm)} \to \ell^2(\mathbb{Z}^{d-1})$ by

$$\mathrm{G}_q^\pm = (\mathcal{H}_q^\pm)^* \mathrm{T}|_{\overline{\mathcal{R}(\mathcal{H}_q^\pm)}} \tag{60}$$

where the operator T is defined by (20).

5.2 Some properties of the operators $\mathcal{H}_q^\pm$, $\mathrm{N}_q^\pm$ and $\mathrm{G}_q^\pm$

We prove in this sections the needed properties in order to establish the theoretical justifications of sampling methods. This Section is somehow the equivalent of Section 3.2 but for the case single Floquet-Bloch mode operator. We recall that we make the assumption that

$$\Omega_M \setminus D \text{ is connected an } \partial\Omega_m \cap \overline{D} = \emptyset \text{ for all } m \in \mathbb{Z}_M^{d-1}.$$

For the support of the perturbation ω we assume that

$$\omega \subset \Omega_{m_0}$$

for some $m_0 \in Z_M$ and
$$\mathcal{D}_p \cap \omega = \emptyset.$$
The latter assumption is important in many aspects of the following proofs and indeed a main perspective of the current work is extend our results to the cases $D_p \cap \omega \neq \emptyset$. The first assumption on ω is not restrictive on the size of the perturbation (since one can increase L) but implies for the inverse problem that one has a priori knowledge on the size of ω.
We define for later use the domain
$$\omega_p := \bigcup_{m-m_0\in\mathbb{Z}_M^{d-1}} \omega + mL \tag{61}$$
which is the union of L periodic copies of ω. In all the following of this section
$$q \in \mathbb{Z}_M^{d-1}$$
is a fixed parameter.

Lemma 5.1. *The operator $\mathcal{H}_q^\pm$ defined by* (57) *is injective and*
$$\overline{\mathcal{R}(\mathcal{H}_q^\pm)} = H^q_{\text{inc}}(D) := \{v \in L^2(D), \quad \Delta v + k^2 v = 0 \text{ in } D \text{ and } v|_{D_p} \text{ is } \alpha_q\text{−quasi-periodic}\} \tag{62}$$

Proof. $\mathcal{H}_q^\pm$ is injective since $\mathcal{H}^\pm$ is injective and I_q is injective. We now prove that $(\mathcal{H}_q^\pm)^*$ is injective on $H^q_{\text{inc}}(D)$. Let $\varphi \in H^q_{\text{inc}}(D)$ and assume $(\mathcal{H}_q^\pm)^*(\varphi) = 0$. We define
$$u(x) := \frac{1}{[\![M]\!]} \int_D \Phi_q(x-y)\varphi(y)\,\mathrm{d}y\,.$$
From the expansion of $\Phi_q(x)$ as in (56) and using the same calculations as for (27) we have that $\widehat{u}^\pm(j) = 0$ for all $j \neq q + M\ell$ and $\widehat{u}^\pm(q+M\ell) = ((\mathcal{H}^\pm)^*(\varphi))\,(q+M\ell) = ((\mathcal{H}_q^\pm)^*(\varphi))(\ell) = 0$. Therefore u has all Rayleigh coefficients equal 0, which implies that
$$u = 0, \quad \text{for } \pm x_d > h.$$
We now observe that for all $y \in D$, $\Delta\Phi_q(\cdot;y) + k^2\Phi_q(\cdot;y) = 0$ in the complement of $D_p \cup \omega_p$. This implies that
$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^d \setminus \{D_p \cup \omega_p\}.$$
Using a unique continuation argument we infer that $u = 0$ in $\Omega \setminus D_p \cup \omega_p$. Therefore, $u \in H^2_0(D_p \cup \omega_p)$ by the regularity of volume potentials.
Since $\varphi|_{D_p}$ and Φ_q are α_q−quasi-periodic functions with period L, then for $m \in \mathbb{Z}_M^{d-1}$,
$$u(x) := \frac{1}{[\![M]\!]} \int_\omega \Phi_q(x;y)\varphi(y)\,\mathrm{d}y + \int_{D_p\cap\Omega_m} \Phi_q(x;y)\varphi(y)\,\mathrm{d}y \quad x \in D_p \cap \Omega_m.$$
We recall that $\Delta\Phi_q(\cdot;y) + k^2\Phi_q(\cdot;y) = -\delta_y$ in Ω_m. Using in addition the fact that $D_p \cap \omega = \emptyset$, we obtain for $m \in \mathbb{Z}_M^{d-1}$,
$$\Delta u(x) + k^2 u(x) = -\varphi(x) \text{ in } D_p \cap \Omega_m. \tag{63}$$
Let us set for $m - m_0 \in \mathbb{Z}_M^{d-1}$
$$\varphi_m(x) := e^{\mathrm{i}\alpha_q\cdot mL}\varphi(x - mL) \text{ for } x \in \omega + mL.$$
Then we have, using the α_q−quasi-periodicity of Φ_q
$$u(x) := \frac{1}{[\![M]\!]} \int_{\omega+mL} \Phi_q(x;y)\varphi_m(y)\,\mathrm{d}y + \frac{1}{[\![M]\!]} \int_D \Phi_q(x;y)\varphi(y)\,\mathrm{d}y \quad x \in \omega + mL$$

with $m - m_0 \in \mathbb{Z}_M^{d-1}$. Using again the fact that $D_p \cap \omega_p = \emptyset$ and $\Delta\Phi_q(\cdot; y) + k^2\Phi_q(\cdot; y) = -\delta_y$ in Ω_m we get

$$\Delta u(x) + k^2 u(x) = -\varphi_m \text{ in } \omega + mL. \tag{64}$$

Now define the function $\tilde{\varphi}$ by

$$\tilde{\varphi} = \varphi \text{ in } D_p \text{ and } \tilde{\varphi} = \varphi_m \text{ in } \omega + mL.$$

Clearly

$$\Delta\tilde{\varphi} + k^2\varphi = 0 \text{ in } D_p \cup \omega_p \text{ with } m - m_0 \in \mathbb{Z}_M^{d-1}.$$

Since $u \in H_0^2(D_p \cup \omega_p)$ we then have

$$\int_D (\Delta u + k^2 u)\overline{\tilde{\varphi}} = 0.$$

This implies according to (63) and (64) that

$$\int_{D_p} |\varphi|^2 \,\mathrm{d}x + [\![M]\!] \int_\omega |\varphi|^2 = 0,$$

which implies $\varphi = 0 \quad \text{in} \quad D_p \cup \omega$. This proves the injectivity of $(\mathcal{H}^\pm)^*$ on $H^q_{\mathrm{inc}}(D)$ and hence proves the Lemma. □

The following two Lemmas are probably the most important results. They form two of the important cornerstones for the justification of the GLSM and the differential imaging functional that we shall define later.

Lemma 5.2. *Assume that Assumptions 2.1 and 3.2 hold. Assume in addition that Assumption 2.1 hold with $n = n_p$. Then the operator $\mathrm{G}_q^\pm : H^q_{\mathrm{inc}}(D) \to \ell^2(\mathbb{Z}^{d-1})$ is injective with dense range.*

Proof. We here give the proof of the lemma for the operator G_q^+ since the proof for the operator G_p^- is similar.
Assume that $G_q^+(f) = 0$ for $f \in H^q_{\mathrm{inc}}(D)$. Let $w \in H^1_\#(\Omega_M^h)$ be the associated scattered field via (10) and consider the decomposition w into

$$w = \frac{1}{[\![M]\!]} \sum_{q' \in \mathbb{Z}_M^{d-1}} w_{q'}, \tag{65}$$

where $w_{q'}$ is $\alpha_{q'}-$quasi-periodic with period L. We recall that

$$\Delta w + k^2 n_p w + k^2(n - n_p)w = k^2(n_p - n)f + k^2(1 - n_p)f. \tag{66}$$

Since $(1 - n_p)f$ is α_q-quasi-periodic with period L and $\omega \subset \Omega_{m_0}$ we obtain that (projecting the latter equation on the $L^2(\Omega_{m_0})$ Fourier basis which is α_q-quasi-periodic with period L)

$$\Delta w_q + k^2 n_p w_q + k^2(n - n_p)w = k^2(1 - n)f \quad \text{in} \quad \Omega_{m_0}. \tag{67}$$

In particular we have that $\Delta w_q + k^2 w_q = 0$ in $\Omega_M \setminus \{D_p \cup \omega_p\}$. Using a similar unique continuation argument as the one at the beginning of the proof of Lemma 5.1 we deduce that

$$w_q = 0 \quad \text{in} \quad \Omega_M \setminus \{D_p \cup \omega_p\}.$$

This implies in particular (since $D_p \cap \omega = \emptyset$) $w_q \in H_0^2(D_p)$ and

$$\begin{cases} \Delta w_q + k^2 n_p w_q &= & k^2(1 - n_p)f & \text{in} & D_p \\ \Delta f + k^2 f &= & 0 & \text{in} & D_p \end{cases}$$

Denoting by ITP(n, D, Φ) the interior transmission problem (23) with $\varphi = \Phi$ and $\psi = \frac{\partial \Phi}{\partial \nu}$, assumption 3.2 implies in particular that ITP$(n_p, D_p, 0)$ is well posed. We then obtain that $f = 0$ in D_p and $w_q = 0$ in D_p. On the other hand, using the fact that $n_p = 1$ in ω we also have $w_q \in H_0^2(\omega)$ and

$$\left\{ \begin{array}{lcll} \Delta w_q + k^2 w_q + k^2(n-1)w &=& k^2(1-n)f & \text{in} \quad \omega. \end{array} \right. \tag{68}$$

The fact that $w_q \in H_0^2(\omega)$ gives for instance that

$$\int_\omega (\Delta w_p + k^2 w_p)\theta = 0, \tag{69}$$

for all $\theta \in H_{inc}(\omega) := \{\theta \in L^2(\omega),\ \Delta\theta + k^2\theta = 0\}$. Now, taking the L^2 scalar product of (68) with θ we arrive at

$$\int_\omega \left(k^2(1-n)f + k^2(1-n)w\right)\overline{\theta} = 0 \tag{70}$$

for all $\theta \in H_{inc}(\omega)$. Recall that the solution w of (10) can be represented as

$$w(x) = \int_{D_p \cup \omega} k^2(n-1)(w+f)(y)\Phi(x;y)\,\mathrm{d}y \quad x \in \Omega_M. \tag{71}$$

Since $y \mapsto \Phi(x;y) \in H_{inc}(\omega)$ for $x \notin \omega$ and $f = 0$ in D_p, we then obtain from (70) that

$$w(x) = \int_{D_p} k^2(1-n_p)w(y)\Phi(x;y)\,\mathrm{d}y \quad \text{for} \quad x \notin \omega. \tag{72}$$

Let us define $\widetilde{w} \in H^2(\omega)$ by

$$\widetilde{w}(x) := \int_{D_p} k^2(1-n_p)w(y)\Phi(x;y)\,\mathrm{d}y \quad x \in \omega.$$

Then obviously $u = w + f$ and f satisfy

$$\left\{ \begin{array}{ll} \Delta u + k^2 n u = 0 & \text{in} \quad \omega \\ \Delta f + k^2 f = 0 & \text{in} \quad \omega \\ u - f = \widetilde{w} & \text{on} \quad \partial\omega \\ \dfrac{\partial(u-f)}{\partial\nu} = \dfrac{\partial\widetilde{w}}{\partial\nu} & \text{on} \quad \partial\omega \end{array} \right. \tag{73}$$

Since

$$\Delta\widetilde{w} + k^2\widetilde{w} = 0 \text{ in } \omega$$

and ITP$(n, \omega, \widetilde{w})$ is well posed (by Assumption 3.2 and the fact that $D_p \cap \omega_p = \emptyset$) we get that

$$f = -\widetilde{w} \quad \text{and} \quad u = 0 \quad \text{in}\ \omega.$$

We then conclude that $w + f = 0$ in ω, i.e. $w = \tilde{w}$ in ω. This implies for instance that

$$w(x) = \int_{D_p} k^2(1-n_p)w(y)\Phi(x;y)\,\mathrm{d}y \quad \text{for} \quad x \notin \Omega_M \tag{74}$$

which means in particular that w is solution to (10) with $n = n_p$ and $f = 0$. We then infer that $w = 0$ by uniqueness of the solution to the scattering problem for $n = n_p$ (third assumption of our Lemma). This proves that $f = 0$ in ω and finishes the arguments for the injectivity of the operator G_q^+.

Using the injectivity we now prove the denseness of the range of $\mathrm{G}_q^{\pm}$. The proof can be done as in Theorem 3.5. Let $g \in \overline{\mathcal{R}(\mathrm{G}_q^+)}^{\perp}$. Then

$$\left(\mathrm{G}_q^+(f), g\right)_{\ell^2(\mathbb{Z}^{d-1})} = 0,\ \forall f \in H^q_{\text{inc}}(D).$$

Consider f of the form $f = \overline{\mathcal{H}_q^+(a)}$ for some $a \in \ell^2(\mathbb{Z}^{d-1})$. Since $\mathrm{G}_q^+ = (\mathcal{H}_q^+)^*\mathrm{T}$, we then have

$$\langle \mathrm{T}(\overline{\mathcal{H}_q^+(a)}), \mathcal{H}_q^+(g)\rangle_{L^2(D)} = 0,\ \forall a \in \ell^2(\mathbb{Z}^{d-1}). \tag{75}$$

Let $w(a)$ and $w(g)$ solution to (10) with respect to $\overline{\mathcal{H}_q^+(a)}$ and $\overline{\mathcal{H}_q^+(g)}$. From Lemma 3.4 we get

$$\begin{aligned}\left(\mathrm{T}(\overline{\mathcal{H}_q^+(a)}), \mathcal{H}_q^+(g)\right)_{L^2(D)} &= k^2 \int_D (n-1)(\overline{\mathcal{H}_q^+ a} + w(a))\, \overline{\mathcal{H}_q^+ g}\, dx \\ &= k^2 \int_D (n-1)(\overline{\mathcal{H}_q^+ g} + w(g))\, \overline{\mathcal{H}_q^+ g}\, dx.\end{aligned}$$

Therefore,

$$\left(\mathrm{T}(\overline{\mathcal{H}_q^+(g)}), \mathcal{H}_q^+(a)\right)_{L^2(D)} = \left(\mathrm{T}(\overline{\mathcal{H}_q^+(a)}), \mathcal{H}_q^+(g)\right)_{L^2(D)}, \quad \forall\, a \in \ell^2(\mathbb{Z}^{d-1}).$$

We deduce from (75) that

$$\left(\mathrm{G}_q^+(\overline{\mathcal{H}_q^+(g)}), a\right)_{\ell^2(\mathbb{Z}^{d-1})} = 0,\ \forall a \in \ell^2(\mathbb{Z}^{d-1}),$$

which implies $\mathrm{G}_q^+(\overline{\mathcal{H}_q^+(g)}) = 0$. The injectivity of G_q^+ ensure that $\mathcal{H}_q^+ g = 0$ in D and consequently $g = 0$ (using Lemma 5.1). This proves the denseness of the range of G_q^+. □

Before continuing with the characterization of $D_p \cup \omega_p$ in terms of the range of $\mathrm{G}_q^{\pm}$ let us make the following simple observations. Let $f \in H_{\text{inc}}(D)$ and let w be the solution of problem (10) associated with f and consider the decomposition

$$w = \frac{1}{[\![M]\!]} \sum_{q \in \mathbb{Z}_M^{d-1}} w_q,$$

as in Lemma 5.2. Let us denote by $a^{\pm}(w) \in \ell^2(\mathbb{Z}^{d-1})$ be the Rayleigh sequences of w. Then for fixed q

$$\begin{cases} \frac{1}{[\![M]\!]} a^{\pm}(w_q)(q + jM) = a^{\pm}(w)(q + jM), & \forall j \in \mathbb{Z}^{d-1} \\ \frac{1}{[\![M]\!]} a^{\pm}(w_q)(\ell) = 0, \quad \forall \ell \neq q + jM. \end{cases}$$

Therefore,

$$\mathrm{I}_q^*(a^{\pm}(w)) = \frac{1}{[\![M]\!]} \mathrm{I}_q^*(a^{\pm}(w_q))$$

(and in particular $\mathrm{I}_q^* \widehat{\Phi}^{\pm}(\cdot; z) = \frac{1}{[\![M]\!]} \mathrm{I}_q^* \widehat{\Phi}_q^{\pm}(\cdot; z)$).

Lemma 5.3. *Under the same assumptions as in Lemma 5.2 we have that* $\mathrm{I}_q^* \widehat{\Phi}_q^{\pm}(\cdot; z) \in \mathcal{R}(\mathrm{G}_q^{\pm})$ *if and only if* $z \in D_p \cup \omega_p$.

Proof. We recall that $\widehat{\Phi}_q^{\pm}(\cdot; z)$ is the Rayleigh sequence of $\Phi_q(\cdot; z)$.
Let $z \in D_p$. We the consider (u_q, v_q) solution of $\mathrm{ITP}(n, D, \Phi_q(\cdot; z))$. Let us define

$$w = \begin{cases} u_q - v_q & \text{in} \quad D_p. \\ \Phi_q & \text{in} \quad \Omega_M \setminus D_p. \end{cases}$$

Then $w \in H^2_{\#,\mathrm{loc}}(\Omega_M)$ and satisfies the scattering problem (10) with $f = v_q$ (remark that $v_q = -\Phi_q$ in ω). Therefore $\mathrm{G}^{\pm}(v_q) = \widehat{\Phi}^{\pm}_q(\cdot; z)$. Applying I^*_q to the two sides of the equality we get $\mathrm{G}^{\pm}_q(v_q) = \mathrm{I}^*_q\widehat{\Phi}^{\pm}_q(\cdot; z)$. Now let $z \in \omega$ and consider $v \in H_{\mathrm{inc}}(D)$ is such that $G(v) = \widehat{\Phi}^{\pm}(\cdot; z)$ as in Lemma. By construction

$$\mathrm{I}^*_q G(Mv) = [\![M]\!]\mathrm{I}^*_q\widehat{\Phi}^{\pm}(\cdot; z) = \mathrm{I}^*_q\widehat{\Phi}^{\pm}_q(\cdot; z) \tag{76}$$

However the function $v \notin H^q_{\mathrm{inc}}(D)$ since the function v is constructed such that $(u, v) \in L^2(D) \times L^2(D)$ is a solution of $\mathrm{ITP}(n, D, \Phi(\cdot; z))$ and therefore $v = -\Phi(\cdot; z)$ in D_p which is not an α_q-quasi-periodic function with period L. We describe in the following how one can construct from v a function $\tilde{v} = H^q_{\mathrm{inc}}(D)$ such that $G_q(v) = G_q(\tilde{v})$. We first decompose $v|_{D_p}$ into

$$v = \frac{1}{[\![M]\!]}\sum_{q' \in \mathbb{Z}^{d-1}_M} v_{q'}$$

where $v_{q'}$ is $\alpha_{q'}-$quasi-periodic function with period L as in 52. Each of the $v_{q'}$ is also a folution to $\Delta v_{q'} + k^2 v_{q'} = 0$ in D_p. Remark that for our case $v_{q'} = \Phi_{q'}(\cdot, z)$. We set $\tilde{v} = v - \frac{1}{[\![M]\!]}v_q$. Now consider the solution $\tilde{w} \in H^2_{\#}(\Omega^h_M)$ to problem (7) with n replaced by n_p and u^i by v. In particular

$$\Delta\tilde{w} + k^2 n_p\tilde{w} = k^2(1 - n_p)\tilde{v} \quad \text{in} \quad \Omega^h_M$$

with the Rayleigh radiation condition. Using the $L^2(\Omega^h_M)$ orthogonality of $(1-n_p)\tilde{v}$ with α_q-quasi-periodic functions and uniqueness the solution of the scattering problem for we simply get that

$$\tilde{w}_q = 0.$$

If we denote by $\tilde{a}^{\pm}$ the Rayleigh sequences associated with $\tilde{w}$, then the latter ensures that $\mathrm{I}^*_q(\tilde{a}^{\pm}) = 0$. Now observe that we can also write the equation for $\tilde{w}$ as

$$\Delta\tilde{w} + k^2 n\tilde{w} = k^2(1 - n_p)\tilde{v} + k^2(n_p - n)(-\tilde{w}) \quad \text{in} \quad \Omega^h_M$$

and since $\omega \cap D_p = \emptyset$, we have $\Delta\tilde{w} + k^2\tilde{w} = 0$ in ω. Therefore the function $\hat{v}$ defined by $\hat{v} = \tilde{v}$ in D_p and $\hat{v} = -\tilde{w}$ in ω satisfies $\hat{v} \in H_{\mathrm{inc}}(D)$ and

$$G^{\pm}(\hat{v}) = \tilde{a}^{\pm}.$$

Therefore $\mathrm{I}^*_q G^{\pm}(\hat{v}) = 0$, $(v - \hat{v}) \in H^q_{\mathrm{inc}}(D)$ and

$$\mathrm{I}^*_q G^{\pm}(v - \hat{v}) = \mathrm{I}^*_q G^{\pm}(v).$$

(Remark that if $\hat{v} \neq 0$ then $G^{\pm}(v - \hat{v}) \neq G^{\pm}(v)$). Applying this procedure to the v in (76) we obtain that, $[\![M]\!](v - \hat{v}) \in H^q_{\mathrm{inc}}(D)$ and

$$G_q([\![M]\!](v - \hat{v})) = \mathrm{I}^*_q\widehat{\Phi}^{\pm}_q(\cdot; z).$$

Now consider the case where $z \in \omega + mL$ with $m - m_0 \in \mathbb{Z}^{d-1}_M$. We recall that $\widehat{\Phi}^{\pm}_q(\cdot; z) = e^{imL\cdot\alpha_q}\widehat{\Phi}^{\pm}_q(\cdot; z - mL)$. Therefore, if we consider $v \in H^q_{\mathrm{inc}}(D)$ such that $G_q(v) = \mathrm{I}^*_q\widehat{\Phi}^{\pm}_q(\cdot; z - mL)$, which is possible by the previous step since $z - mL \in \omega$, then

$$G_q(e^{\mathrm{i}mL\cdot\alpha_q}v) = \mathrm{I}^*_q(\widehat{\Phi}^{\pm}_q(\cdot; z)).$$

We finally consider the case $z \notin D_p \cup \omega_p$. If $G_q(v) = \mathrm{I}^*_q\widehat{\Phi}^{\pm}_q(\cdot; z)$, then using the same unique continuation argument as in the proof of Lemma 5.2 we get that $w_q = \Phi_q$ in $\Omega_M \setminus \{D_p \cup \omega_p\}$ where w is the solution of (10) with $f = v$ and w_q is defined by (65). This gives a contradiction since w_q is locally H^2 in $\Omega_M \setminus \{D_p \cup \omega_p\}$) while $\Phi_q(\cdot; z)$ is not. □

The following Lemma allows us to infer some reconstruction results for the domain $D_p \cup \omega_p$ using the Factorization method but is not important for the differential imaging functional we shall introduce later.

Lemma 5.4. $\mathrm{I}_q^* \widehat{\Phi}_q^{\pm}(\cdot; z) \in \mathcal{R}((\mathcal{H}_q^{\pm})^*)$ *if and only if* $z \in D_p \cup \omega_p$.

Proof. We first consider $z \in D_p \cup \omega_p$. We define ρ a C^∞ cut-off function such that

$$\begin{aligned} \rho = 1 \quad &\text{in} \quad \Omega \setminus D_p \cup \omega_p \\ \rho = 0 \quad &\text{in a neighborhood of } \{z + mL,\, m \in \mathbb{Z}_M^{d-1}\} \end{aligned} \tag{77}$$

and ρ is periodic with period L. We set $v := \rho\, \Phi_q(\cdot; z)$. Obviously v is α_q-quasi-periodic with period L and

$$\widehat{v}^{\pm}(j) = \widehat{\Phi}_q^{\pm}(j; z).$$

Moreover, since $f := -(\Delta v + k^2 v)$ is α_q-quasi-periodic and with support in $D_p \cup \omega_p$ then

$$v = \frac{1}{[\![M]\!]} \int_{D_p \cup \omega_p} f(y) \Phi_q(x - y)\, \mathrm{d}y$$

Obviously,

$$\widehat{v}^{\pm}(q + Mj) = \frac{1}{[\![M]\!]} \int_{D_p \cup \omega_p} f(y) \overline{u^{i,\pm}}(y; q + Mj) dy.$$

We see that

$$\int_{\omega_p} f(y) \overline{u^{i,\pm}}(y; q + Mj) dy = [\![M]\!] \int_{\omega} f(y) \overline{u^{i,\pm}}(y; q + Mj) dy$$

since f and $u^{i,\pm}(\cdot; q + Mj)$ are α_q-quasi-periodic functions with period L. Therefore,

$$\widehat{v}^{\pm}(q + Mj) = \frac{1}{[\![M]\!]} \int_{D_p} f(y) \overline{u^{i,\pm}}(y; q + Mj) dy + \int_{\omega} f(y) \overline{u^{i,\pm}}(y; q + Mj) dy.$$

We then obtain $\mathrm{I}_q^* \widehat{v}^{\pm} = \mathrm{I}_q^* (\mathcal{H}^{\pm})^* \varphi$ with

$$\varphi := \begin{cases} \frac{1}{[\![M]\!]} f & \text{in } D_p \\ f & \text{in } \omega. \end{cases}$$

Therefore

$$(\mathcal{H}_q^{\pm})^* \varphi = \mathrm{I}_q^* \widehat{\Phi}_q^{\pm}(\cdot; z)$$

If $z \notin D_p \cup \omega_p$, using the same unique continuation argument as in the proof of Lemma 5.1 we see that if $(\mathcal{H}_q^{\pm})^* \varphi = \mathrm{I}_q^* \widehat{\Phi}_q^{\pm}(\cdot; z)$ then the function defined by

$$w(x) = \int_D \Phi(x - y) \varphi(y)\, \mathrm{d}y$$

verifies $w_q = \Phi_q(\cdot; z)$ in $\Omega \setminus D_p \cup \omega_p \cup \{z\}$ where w_q is defined as in (65). This gives a contradiction since w_q is smooth in a neighborhood of z but $\Phi_q(\cdot; z)$ is singular at z. We then get the contradiction if $z \notin D_p \cup \omega_p$. $\square$

We now end this section with direct applications of the sampling methods that have been introduced in Section 4 to the case where one uses the operator $\mathrm{N}_q^{\pm}$. The previous technical results allow us to phrase theorems related to reconstructing the domain $D_p \cup \omega_p$. The first one is related to the factorization method that employs the operator $\mathrm{N}_{q,\sharp}$ that we shall define here as

$$\mathrm{N}_{q,\sharp}^{\pm} := \mathrm{I}_q^* \mathrm{N}_{\sharp}^{\pm} \mathrm{I}_q$$

The proof of the following theorem is based on the same arguments as the proof of Theorem 4.2 and is a consequence of the factorization 59, the properties of the operator T formulated in Lemma 3.6 and the technical Lemma 5.4.

Theorem 5.5. *Under the same assumptions of Lemma 3.6 we have that*

$$\mathrm{N}^{\pm}_{q,\sharp} = (\mathcal{H}^{\pm}_q)^* \, \mathrm{T}^{\pm}_{\sharp} \, \mathcal{H}^{\pm}_q, \tag{78}$$

where $\mathrm{T}^{\pm}_{\sharp} : L^2(D) \to L^2(D)$ *is the same as in Theorem 4.2. In particular it is self-adjoint and coercive on* $H^q_{\mathrm{inc}}(D)$. *Moreover,* $\mathrm{I}^*_q \widehat{\Phi}^{\pm}_q(\cdot; z) \in \mathcal{R}((\mathrm{N}^{\pm}_{q,\sharp})^{1/2})$ *if and only if* $z \in D_p \cup \omega_p$.

Proof. The proof of the factorization (78) follows directly from (41). The second part of the theorem follows from the technical Lemma 5.4 and Theorem A.2. □

We remark that we could also have defined $\mathrm{N}^{\pm}_{q,\sharp}$ as in the definition of $N^{\pm}_{\sharp}$ and derived the same results. However, proving in this way that the middle operator $\mathrm{T}^{\pm}_{\sharp}$ is the same as for $N^{\pm}_{\sharp}$ would have been less straightforward.

The second useful corollary is the application of the GLSM algorithm. Indeed the following theorem can be proved in a similar way as Theorem 4.4 using Theorem 5.5, and the technical Lemmas 5.1, 5.2 and 5.3.

Theorem 5.6. *Assume that Assumptions of Lemmas 3.6 and 5.2 hold. Then the results of 4.4 are 4.5 still true if* D *is replaced by* $D_p \cup \omega_p$*, the operators* $\mathcal{H}^{\pm}$, $\mathrm{G}^{\pm}$, $\mathrm{N}^{\pm}$ *and* $\mathrm{N}^{\pm}_{\sharp}$ *are replaced by the operators* $\mathcal{H}^{\pm}_q$, $\mathrm{G}^{\pm}_q$, $\mathrm{N}^{\pm}_q$ *and* $\mathrm{N}^{\pm}_{q,\sharp}$ *respectively and* $\widehat{\Phi}^{\pm}(\cdot; z)$ *is replaced by* $\mathrm{I}^*_q \widehat{\Phi}^{\pm}_q(\cdot; z)$.

5.3 A new differential imaging functional

We now have all the ingredients to introduce the new differential imaging functional for retrieving the domain ω using the measurement operator N^+ or N^-. Compared to the algorithm presented in [5] there is no need here for a measure of the background operator. The latter will be replaced by using the operator N^+_q (or respectively N^-_q) for a fixed $q \in \mathbb{Z}^{d-1}_M$. In addition to the assumptions on the geometry that was made in the previous section we assume that for all $z \in \omega$ there exists $m \in \mathbb{Z}^{d-1}$ such that $z + mL \in \Omega_M \setminus D$. This means in particular that $M > 1$ and therefore there is a least one period that does not contain a defect.

We hereafter assume that the hypothesis of Theorems 5.5 and 5.6 are verified. In order to simplify the notations we only present the results for the operator N^+. We obtain exactly the same results by changing the exponent $+$ with the exponent $-$. We then consider for ϕ and a in $\ell^2(\mathbb{Z}^{d-1})$ the cost functionals

$$\begin{aligned} J^+_\alpha(\phi, a) &:= \alpha(\mathrm{N}^+_\sharp a, a) + \|\mathrm{N}^+ a - \phi\|^2, \\ J^+_{\alpha,q}(\phi, a) &:= \alpha(\mathrm{N}^+_{q,\sharp} a, a) + \|\mathrm{N}^+_q a - \phi\|^2. \end{aligned} \tag{79}$$

We remark that we also have

$$J^+_{\alpha,q}(\phi, a) = \alpha(\mathrm{N}^+_\sharp \mathrm{I}_q a, \mathrm{I}_q a) + \|\mathrm{N}^+_q a - \phi\|^2.$$

Let $a^{\alpha,z}$, $a^{\alpha,z}_q$ and $\tilde{a}^{\alpha,z}_q$ be in $\ell(\mathbb{Z}^{d-1})$ verifying

$$\begin{aligned} J^+_\alpha(\widehat{\Phi}^{\pm}(\cdot; z), a^{\alpha,z}) &\le \inf_{a \in \ell^2(\mathbb{Z}^{d-1})} J^+_\alpha(\widehat{\Phi}^{\pm}(\cdot; z), a) + c(\alpha) \\ J^+_\alpha(\widehat{\Phi}^{\pm}_q(\cdot; z), a^{\alpha,z}_q) &\le \inf_{a \in \ell^2(\mathbb{Z}^{d-1})} J^+_\alpha(\widehat{\Phi}^{\pm}_q(\cdot; z), a) + c(\alpha) \\ J^+_{\alpha,q}(\mathrm{I}^*_q \widehat{\Phi}^{\pm}_q(\cdot; z), \tilde{a}^{\alpha,z}_q) &\le \inf_{a \in \ell^2(\mathbb{Z}^{d-1})} J^+_{\alpha,q}(\mathrm{I}^*_q \widehat{\Phi}^{\pm}_q(\cdot; z), a) + c(\alpha) \end{aligned} \tag{80}$$

with $\frac{c(\alpha)}{\alpha} \to 0$ as $\alpha \to 0$. We then consider the following imaging functional to characterize ω:

$$\mathcal{I}^+_\alpha(z) = \left((\mathrm{N}^+_\sharp a^{\alpha,z}, a^{\alpha,z}) \left(1 + \frac{(\mathrm{N}^+_\sharp a^{\alpha,z}, a^{\alpha,z})}{D^+(a^{\alpha,z}_q, \tilde{a}^{\alpha,z}_q)} \right) \right)^{-1} \tag{81}$$

where for a and b in $\ell^2(\mathbb{Z}^{d-1})$,

$$D^+(a, b) := \left(\mathrm{N}^+_\sharp (a - \mathrm{I}_q b), (a - \mathrm{I}_q b) \right).$$

Theorem 5.7. *Under the assumptions of Theorem 5.6, we have that*

$$z \in \omega \text{ if and only if } \lim_{\alpha\to 0} \mathcal{I}_\alpha^+(z) > 0.$$

Proof. If $z \notin D$ then from Theorem 4.4 applied to N^+ we get that $(\mathrm{N}_\sharp^+ a^{\alpha,z}, a^{\alpha,z}) \to +\infty$ as $\alpha \to 0$ and therefore $\lim\limits_{\alpha\to 0} \mathcal{I}_\alpha(z) = 0$.

If $z \in D_p$, we remark that according to Theorems 4.9 and 5.6, when $\alpha \to 0$, the sequences $\mathcal{H}^+ a_q^{\alpha,z}$ and $\mathcal{H}_q^+ \tilde{a}_q^{\alpha,z}$ respectively converge in $L^2(D)$ to $v \in H_{\text{inc}}(D)$ and $\tilde{v} \in H_{\text{inc}}^q(D)$ such that

$$G(v) = \widehat{\Phi}_q^+(\cdot; z) \text{ and } G_q(\tilde{v}) = \mathrm{I}_q^* \widehat{\Phi}_q^+(\cdot; z).$$

According to the proof of Theorem 3.5 and Lemma 5.3, the functions v and $\tilde{v}$ are solutions to $\mathrm{ITP}(n, D, \Phi_q(\cdot; z))$ and therefore are the same. From the factorization of $\mathrm{N}_\sharp^+$ and the definition of $\mathcal{H}_q^+$ we get

$$D^+(a_q^{\alpha,z}, \tilde{a}_q^{\alpha,z}) = \big(\mathrm{T}_\sharp(\mathcal{H}^+ a_q^{\alpha,z} - \mathcal{H}_q^+ \tilde{a}_q^{\alpha,z}), \mathcal{H}^+ a_q^{\alpha,z} - \mathcal{H}_q^+ \tilde{a}_q^{\alpha,z}\big)$$

and therefore

$$D^+(a_q^{\alpha,z}, \tilde{a}_q^{\alpha,z}) \leq \|\mathrm{T}_\sharp\| \|\mathcal{H}^+ a_q^{\alpha,z} - \mathcal{H}_q^+ \tilde{a}_q^{\alpha,z}\|^2_{L^2(D} \to 0, \quad \text{as } \alpha \to 0.$$

From Theorem 4.4, $(\mathrm{N}_\sharp a^{\alpha,z}, a^{\alpha,z})$ converges as $\alpha \to 0$ to $(\mathrm{T}_\sharp v_0, v_0) < \infty$ where $v_0 \neq 0$ is the solution of $G(v_0) = \widehat{\Phi}_q^+(\cdot; z)$. We then conclude that

$$\lim_{\alpha\to 0} \mathcal{I}_\alpha^+(z) = 0 \text{ if } z \in D_p.$$

Finally, if $z \in \omega$ then again by Theorem 4.4, $(\mathrm{N}_\sharp a^{\alpha,z}, a^{\alpha,z}) < +\infty$ converges as $\alpha \to 0$ to $(\mathrm{T}_\sharp v_0, v_0) < \infty$ where $v_0 \neq 0$ is the solution of $G(v_0) = \widehat{\Phi}_q^+(\cdot; z)$. However,

$$D^+(a_q^{\alpha,z}, \tilde{a}_q^{\alpha,z}) \to \infty, \quad \alpha \to 0$$

since by Theorem 4.9 $(\mathrm{N}_\sharp^+ a_q^{\alpha,z}, a_q^{\alpha,z}) \to +\infty$ while $(\mathrm{N}_\sharp^+ \mathrm{I}_q \tilde{a}_q^{\alpha,z}, \mathrm{I}_q \tilde{a}_q^{\alpha,z}) \to (T\tilde{v}, \tilde{v}) < +\infty$ where $\tilde{v} \in H_{\text{inc}}^q(D)$ is such that $G_q(\tilde{v}) = \mathrm{I}_q^* \widehat{\Phi}_q^+(\cdot; z)$. Therefore,

$$\lim_{\alpha\to 0} \mathcal{I}_\alpha^+(z) \neq 0 \text{ if } z \in \omega$$

and the Theorem is proved. □

Let us conclude this section by indicating that indeed, following Theorem 4.5 in the case of noisy measurements, one has to redefine the functionals J_α^+ and $J_{\alpha,q}^+$ as

$$\begin{aligned} J_\alpha^{+,\delta}(\phi, a) &:= \alpha\left((\mathrm{N}_\sharp^{+,\delta} a, a) + \delta\alpha^{-\eta} \|\mathrm{N}_\sharp^{+,\delta}\| \|a\|^2\right) + \|\mathrm{N}^{+,\delta} a - \phi\|^2, \\ J_{\alpha,q}^{+,\delta}(\phi, a) &:= \alpha\left((\mathrm{N}_\sharp^{+,\delta} \mathrm{I}_q a, \mathrm{I}_q a) + \delta\alpha^{-\eta} \|\mathrm{N}_\sharp^{+,\delta}\| \|a\|^2\right) + \|\mathrm{N}_q^{+,\delta} a - \phi\|^2 \end{aligned} \tag{82}$$

where $\eta < 1$ is a fixed positive parameter and δ the relative noise level. We then consider $a_\delta^{\alpha,z}$, $a_{q,\delta}^{\alpha,z}$ and $\tilde{a}_{q,\delta}^{\alpha,z}$ in $\ell(\mathbb{Z}^{d-1})$ as the minimizers of respectively

$$J_\alpha^{+,\delta}(\widehat{\Phi}^+(\cdot; z), a), \; J_\alpha^{+,\delta}(\widehat{\Phi}_q^+(\cdot; z), a) \text{ and } J_{\alpha,q}^{+,\delta}(\widehat{\Phi}_q^+(\cdot; z), a).$$

We define the indicator function in the noisy case as

$$\mathcal{I}_\alpha^{+,\delta}(z) = \left(\mathcal{G}^{+,\delta}(a_\delta^{\alpha,z}) \left(1 + \frac{\mathcal{G}^{+,\delta}(a_\delta^{\alpha,z})}{D^{+,\delta}(a_{q,\delta}^{\alpha,z}, \tilde{a}_{q,\delta}^{\alpha,z})}\right)\right)^{-1} \tag{83}$$

where for a and b in $\ell^2(\mathbb{Z}^{d-1})$,

$$D^{+,\delta}(a,b) := \Big(\mathrm{N}_\sharp^{+,\delta}(a - \mathrm{I}_q b), (a - \mathrm{I}_q b)\Big)$$

and

$$\mathcal{G}^{+,\delta}(a) := (\mathrm{N}_\sharp^{+,\delta} a, a) + \delta\alpha^{-\eta}\|\mathrm{N}_\sharp^{+,\delta}\|\|a\|^2.$$

Then following the lines of the previous proof one can prove (thanks to Theorem 4.5 and equivalent version for N_q^+)

Theorem 5.8. *Under the assumptions of Theorem 5.6, we have that*

$$z \in \omega \text{ if and only if } \lim_{\alpha\to 0} \liminf_{\delta\to 0} \mathcal{I}_\alpha^{+,\delta}(z) > 0.$$

6 Validating Numerical Experiments

We here give some numerical examples for testing the sampling methods introduced in Section 4 and the sampling method using a single Floquet-Bloch mode introduced in Section 5. We only consider here two dimensional examples. In order to obtain the data for the inverse problem, we consider N_{inc} incident down-to-up or up-to-down plane-waves given by formula (13) or (14). The synthetic data has been generated by solving the forward scattering problem using the spectral discretization scheme of the volume integral formulation of the problem presented in [20].
We denote by

$$\mathbb{Z}_{inc}^{d-1} := \{j = q + M\ell,\ q \in \mathbb{Z}_M^{d-1},\ \ell \in \mathbb{Z}^{d-1} \text{ and } \ell \in [\![-N_{min}, N_{max}]\!]\}$$

the set of indices for the incident waves (which is also the set of indices for measured Rayleigh coefficients). The values of all parameters will be indicated below. The discrete version of the operators $\mathrm{N}^\pm$ are given by the matrices of size $N_{inc} \times N_{inc}$,

$$\mathrm{N}^\pm := \Big(\widehat{u^s}^\pm(\ell; j)\Big)_{\ell, j \in \mathbb{Z}_{inc}^{d-1}}. \tag{84}$$

We add some noise to the data as:

$$\mathrm{N}^{\pm,\delta}(j,\ell) := \mathrm{N}^\pm(j,\ell)\big(1 + \delta A(j,\ell)\big), \quad \forall (j,\ell) \in \mathbb{Z}_{inc}^{d-1} \times \mathbb{Z}_{inc}^{d-1} \tag{85}$$

where $A = (A(j,\ell))_{N_{inc}\times N_{inc}}$ is a matrix of uniform complex random variables with real and imaginary parts in $[-1,1]^2$ and $\delta > 0$ is the noise level. We use in all following examples $\delta = 1\%$.

Let $a_{\alpha,G}^\pm(z) \in \ell^2(\mathbb{Z}^{d-1})$ be solution of (49) with the parameter α defined in (50). We define

$$\mathcal{I}_+^{\mathrm{GLSM}}(z) = \frac{1}{|(N_\sharp^{+,\delta} a_{\alpha,G}^+(z), a_{\alpha,G}^+(z))| + \delta\|N_\sharp^{+,\delta}\|\|a_{\alpha,G}^+(z)\|^2}$$
$$\mathcal{I}_-^{\mathrm{GLSM}}(z) = \frac{1}{|(N_\sharp^{-,\delta} a_{\alpha,G}^-(z), a_{\alpha,G}^-(z))| + \delta\|N_\sharp^{-,\delta}\|\|a_{\alpha,G}^-(z)\|^2}$$

and the indicator function

$$\mathcal{I}^{\mathrm{GLSM}}(z) := \mathcal{I}_+^{\mathrm{GLSM}}(z) + \mathcal{I}_-^{\mathrm{GLSM}}(z)$$

that we shall use later.

Let $q \in \mathbb{Z}_M^{d-1}$. Let $a_{\alpha,q,G}^\pm(z) \in \ell^2(\mathbb{Z}^{d-1})$ be defined similar to $a_{\alpha,G}^\pm(z)$ with $\widehat{\Phi}^\pm(\cdot; z)$ is replaced by $\widehat{\Phi}_q^\pm(\cdot; z)$. We then similarly define $\mathcal{I}_\pm^{\mathrm{GLSMq}}(z)$ and then $\mathcal{I}^{\mathrm{GLSMq}}(z)$

In addition to the indicator function $\mathcal{I}^{+,\delta}(z)$ defined by (83) and the similarly defined indicator function $\mathcal{I}^{-,\delta}(z)$, we also define

$$\mathcal{I}^\delta(z) := \mathcal{I}^{+,\delta}(z) + \mathcal{I}^{-,\delta}(z).$$

Example 1:

In the first example, we restrict ourselves to a simple setting for the geometry represented in the left of Figure 2. The background geometry D_p is constituted by discs of radii r_1, r_2 and the geometry of the perturbation is a disc of radius r_w. The physical parameters are set as

$$k = \pi/3.14,\ n_p = 2 \text{ in } D_p \text{ and } \ n = 4 \text{ in } \omega. \tag{86}$$

Set $\lambda := 2\pi/k$ as the wavelength. Then the geometrical parameters are

$$L = \pi\lambda,\ h = 1.5\lambda,\ r_1 = 0.2\lambda,\ r_2 = 0.3\lambda \text{ and } r_w = 0.2\lambda. \tag{87}$$

The coordinate system is chosen such that the center of one disc of D_p is $(0, 0.8\lambda)$ and the center of ω is $(1.2\lambda, \lambda)$. Finally we choose as parameters for the discrete model

$$M = 2,\ N_{min} = 5 \text{ and } N_{max} = 5. \tag{88}$$

Figure 2-right shows the reconstruction of the domain D. The reconstruction of the domain ω is given in Figure 3-left where one observes that it is indeed isolated from the background structure. In this example we observe that the defect could have been identified using 2-right and 3-right that corresponds to the reconstruction of the background geometry.

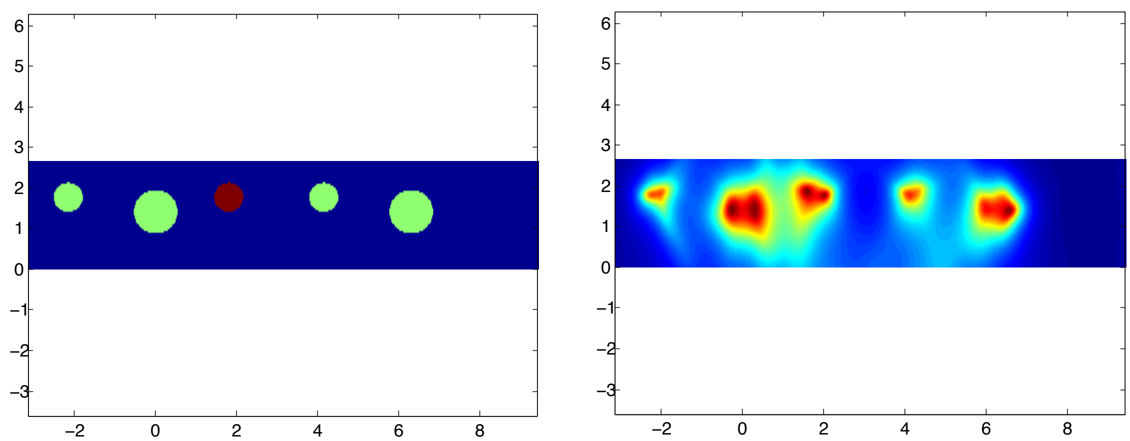

Figure 2: Left: The exact geometry for Example 1. Right: the reconstruction of this domain using $z \mapsto \mathcal{I}^{\mathrm{GLSM}}(z)$.

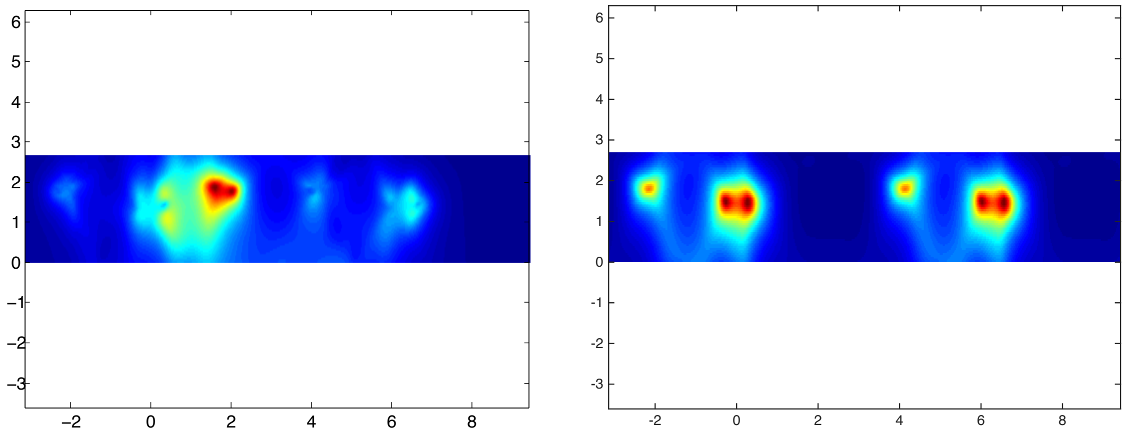

Figure 3: Exact geometry is given in Figure 2. Left: The reconstruction of the local perturbation using $z \mapsto \mathcal{I}^{\delta}(z)$. Right: the reconstruction of periodic background using $z \mapsto \mathcal{I}^{\mathrm{GLSMq}}(z)$ with $q = 1$.

Example 2.

In this example we consider a more complicated background geometry as depicted in Figure 4-left. The physical parameters and the parameters for collecting the data are the same as in the previous example. The geometry of the perturbation is also the same. We observe that simultaneous reconstruction of the background and the perturbation as given in Figure 4-right does not allow to have a clear identification of the defect, even if a combination with Figure 5-right is used. However, the reconstruction of the geometry of the perturbation using the differential indicator function as shown in Figure 5-left still provides a similar accuracy as in the first example.

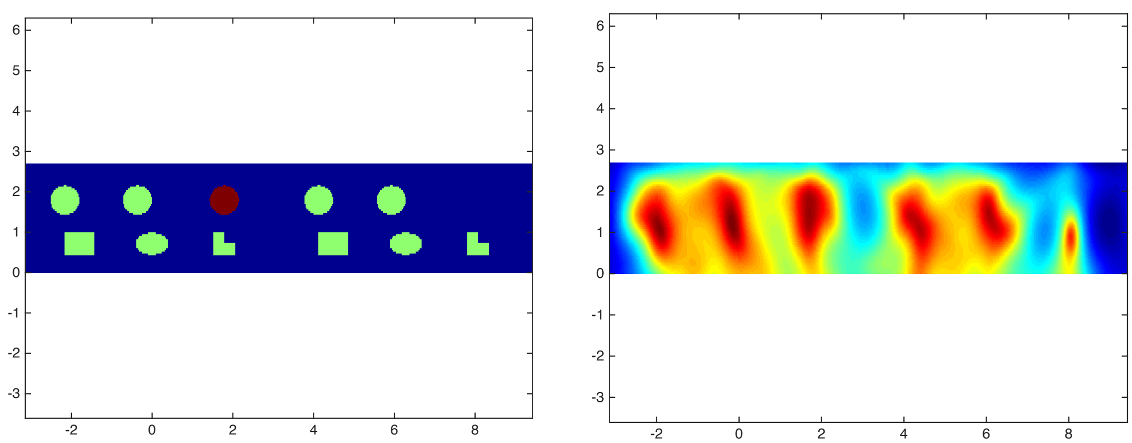

Figure 4: Left: The exact geometry for Example 2. Right: the reconstruction of this domain using $z \mapsto \mathcal{I}^{\mathrm{GLSM}}(z)$.

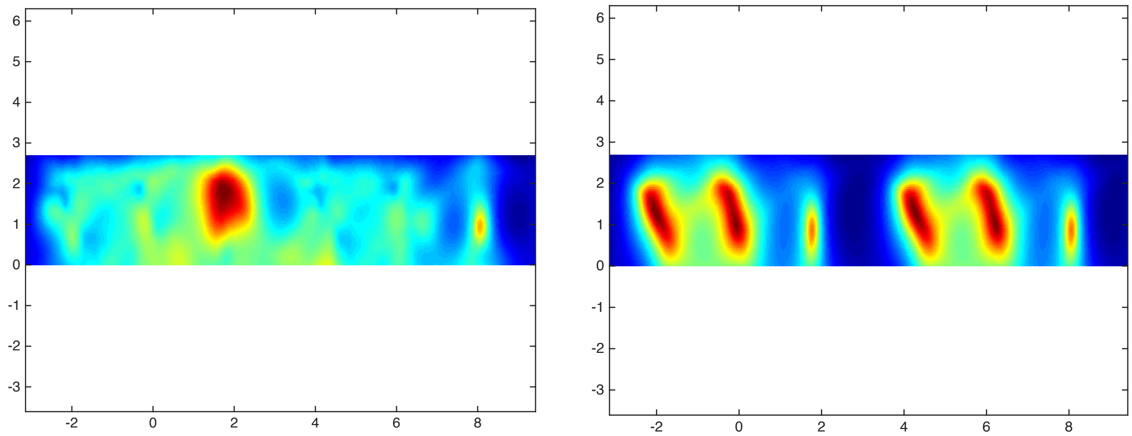

Figure 5: Exact geometry is given in Figure 4. Left: The reconstruction of the local perturbation using $z \mapsto \mathcal{I}^{\delta}(z)$. Right: the reconstruction of periodic background using $z \mapsto \mathcal{I}^{\mathrm{GLSMq}}(z)$ with $q = 1$.

A Abstract theoretical foundations of the sampling methods

We summarize in this appendix the main theoretical results of the literature that allows us to establish the sampling methods for the locally perturbed periodic domains. We here follow the formulations of the theorems as given in [11] with obvious simplifications in the case of Hilbert spaces which is sufficient for our problem.

A.1 Main theorem for the $F_\sharp$ method

This method is one of two versions of the Factorization method, which was introduced by A.Kirsch in [24] and detailed in [25]. Consider a separable Hilbert space X and an operator $\mathrm{N} : X \to X$ such that factorization

$$\mathrm{N} = (\mathcal{H})^* \mathrm{T} \mathcal{H} \tag{89}$$

where $\mathcal{H} : X \to Y$ and $\mathrm{T} : Y \to Y$ are bounded operators and Y is also a separable Hilbert space. In the following we shall denote by $(\cdot, \cdot)$ the scalar product and by $\|\cdot\|$ the associated norm in X and use the same notation for Y since there is no risk of confusion.

We first make the following assumption on the operator T.

Assumption A.1. *We assume that operator* $\mathrm{T} : Y \to Y$ *satisfies*

$$\operatorname{Im}(\mathrm{T}\varphi, \varphi) \geq 0 \quad \text{or} \quad \operatorname{Im}(\mathrm{T}\varphi, \varphi) \leq 0 \tag{90}$$

for all $\varphi \in \overline{\mathcal{R}(\mathcal{H})}$, $\operatorname{Re}\mathrm{T} = \mathrm{T}_0 + C$ *where* C *is compact on* $\overline{\mathcal{R}(\mathcal{H})}$ *and*

$$(\mathrm{T}_0\varphi, \varphi) \geq \alpha \|\varphi\|^2 \tag{91}$$

for all $\varphi \in \overline{\mathcal{R}(\mathcal{H})}$ *and some* $\alpha > 0$. *Moreover, we assume that one of the following assumptions holds:*

(i) T *is injective on* $\overline{\mathcal{R}(\mathcal{H})}$;

(ii) $\operatorname{Im}(\mathrm{T})$ *is injective on* $\overline{\mathcal{R}(\mathcal{H})} \cap \ker \operatorname{Re}\mathrm{T}$.

Define the opertaor $\mathrm{N}_\sharp$ as

$$\mathrm{N}_\sharp := \frac{1}{2}|\mathrm{N} + \mathrm{N}^*| + \frac{1}{2}|\mathrm{N} - \mathrm{N}^*|.$$

Then the main theorem for the factorization method can be formulated as:

Theorem A.2. *Let* N *be given by* (89) *and assume that* $H : X \to Y$ *is compact and injective and that* T *satisfies Assumption A.1. Then*

$$\mathrm{N}_\sharp = (\mathcal{H})^* \mathrm{T}_\sharp \mathcal{H} \tag{92}$$

where $\mathrm{T}_\sharp : Y \to Y$ *is self-adjoint and satisfies the coercivity property on* $\overline{\mathcal{R}(\mathcal{H})}$

$$(\mathrm{T}_\sharp \varphi, \varphi) \geq \alpha \|\varphi\|^2 \quad \forall\, \varphi \in \overline{\mathcal{R}(\mathcal{H})}.$$

Moreover, the ranges $\mathcal{R}((\mathcal{H})^*)$ *and* $\mathcal{R}((\mathrm{N}_\sharp)^{1/2})$ *coincide.*

A.2 Theoretical foundations of GLSM

This method has been introduced in [4, 6] and we provide here the main theorems that are used in our case. They are mainly extracted from [5, 11]. We consider two separable Hilbert spaces X and Y (as in the previous section) and two operators Let $\mathrm{N} : X \to X$ and $\mathrm{B} : X \to X$, which have the following factorizations

$$\mathrm{N} = \mathrm{G}\,\mathcal{H} \quad \text{ and } \quad \mathrm{B} = (\mathcal{H})^*\, \mathrm{T}_\sharp\, \mathcal{H}. \tag{93}$$

where $\mathcal{H} : X \to Y$ and $\mathrm{T} : Y \to Y$ are bounded operators and where

$$\mathrm{G} : \overline{\mathcal{R}(\mathcal{H})} \mapsto X$$

is also bounded. We consider first the GLSM for noisy free operators. Let $\alpha > 0$ be a given parameter and $\phi \in X$. We define the functional $J_\alpha(\phi; \cdot) : X \to \mathbb{R}$ by

$$J_\alpha(\phi; a) := \alpha |(\mathrm{B}a, a)| + \|\mathrm{N}a - \phi\|^2 \quad \forall a \in X \tag{94}$$

and set

$$j_\alpha(\phi) := \inf_{g \in X} J_\alpha(\phi; g). \tag{95}$$

The first main theorem for GLSM is the following:

Theorem A.3. *Assume that*

- G *is compact and* N *has dense range.*
- $\mathrm{T}_\sharp$ *satisfies the coercivity property*

$$|(\mathrm{T}_\sharp \varphi, \varphi)| > \mu\, \|\varphi\|^2 \quad \forall\, \varphi \in \mathcal{R}(\mathcal{H}), \tag{96}$$

where $\mu > 0$ *is a constant independent of* φ. *Let* $c(\alpha)$ *such that* $c(\alpha)/\alpha \to 0$ *as* $\alpha \to 0$ *and* $\phi \in X$, *an element* $a_\alpha \in X$ *such that*

$$J_\alpha(\phi; a_\alpha) \leq j_\alpha(\phi) + c(\alpha). \tag{97}$$

Then the following holds.

- *If* $\phi \in \mathcal{R}(G)$ *then* $\limsup_{\alpha \to 0} |(\mathrm{B}a, a)| < \infty$.
- *If* $\phi \notin \mathcal{R}(G)$ *then* $\liminf_{\alpha \to 0} |(\mathrm{B}a, a)| = \infty$.

In the previous theorem nothing is said on the convergence or not of $(\mathrm{B}a, a)$. The latter is possible if the operator $\mathrm{T}_\sharp$ is seladjoint (meaning that B is also selfadjoint).

Theorem A.4. *We assume, in addition to the hypothesis of Theorem A.3, that* N *is injective and that* $\mathrm{T}_\sharp$ *is selfadjoint. Consider for* $\alpha > 0$ *and* $\phi \in X$, $a_\alpha \in X$ *such that*

$$J_\alpha(\phi; a_\alpha) \leq j_\alpha(\phi) + c(\alpha) \tag{98}$$

where $\frac{c(\alpha)}{\alpha} \to 0$ *as* $\alpha \to 0$. *Then* $\phi \in \mathcal{R}(G)$ *if and only if* $\lim_{\alpha \to 0} (\mathrm{B}a_\alpha, a_\alpha) < \infty$. *Moreover, in the case* $\phi = G\varphi$, *the sequence* $\mathcal{H}a_\alpha$ *strongly converges to* φ *in* Y *as* α *goes to zero.*

Remark A.5. In the case $B = F$, one can avoid the extra assumption on the operator $\mathrm{T}_\sharp$ in Theorem A.4 by replacing the cost functional J_α with

$$J_\alpha(\phi; g) := \alpha|(\mathrm{N}g, g)| + \alpha^{1-\eta}|(\mathrm{N}g - \phi, g)| + \|\mathrm{N}g - \phi\|^2 \quad \forall g \in X, \tag{99}$$

with $\eta \in]0, 1]$ being a fixed parameter. This type of functional is more suited for the case of limited aperture data which is not considered in our case.

We now state the RGLSM theorem for the case of noisy data. More precisely, we shall assume that one has access to two noisy operators B^δ and N^δ such that

$$\left\|\mathrm{N}^\delta - \mathrm{N}\right\| \le \delta \quad \text{ and } \left\|\mathrm{B}^\delta - \mathrm{B}\right\| \le \delta$$

for some $\delta > 0$. We also assume that the operators B, B^δ N^δ and N are compact. We then consider for $\alpha > 0$ and $\phi \in X^*$ the functional $J_\alpha^\delta(\phi; \cdot) : X \to \mathbb{R}$ defined by

$$J_\alpha^\delta(\phi; a) := \alpha(|(\mathrm{B}^\delta a, a)| + \delta\alpha^{-\eta} \|a\|^2) + \left\|\mathrm{N}^\delta a - \phi\right\|^2 \quad \forall\, a \in X, \tag{100}$$

where $\eta < 1$ is a fixed positive parameter.

Theorem A.6. *Assume that the assumptions of Theorem A.4 hold. Let a_α^δ be the minimizer of $J_\alpha^\delta(\phi; \cdot)$ (defined by (100)) for $\alpha > 0$, $\delta > 0$ and $\phi \in X^*$. Then $\phi \in \mathcal{R}(G)$ if and only if $\lim\limits_{\alpha\to0} \limsup\limits_{\delta\to0} |(\mathrm{B}^\delta a_\alpha^\delta, a_\alpha^\delta)| + \delta\alpha^{-\eta} \left\|a_\alpha^\delta\right\|^2 < \infty$. Moreover, in the case $\phi = G\varphi$, $\lim\limits_{\alpha\to0} \limsup\limits_{\delta\to0} \|\mathcal{H}a_\alpha - \varphi\| = 0$.*

Remark A.7. If B^δ is a positive selfadjoint operator one can directly compute the minimizer a_α^δ of $J_\alpha^\delta(\phi; \cdot)$ (defined by (100)) for $\alpha > 0$, $\delta > 0$ and $\phi \in X^*$ as the solution of

$$(\alpha\mathrm{B}^\delta + \alpha\delta I + (\mathrm{N}^\delta)^*\mathrm{N}^\delta)a_\alpha^\delta = (\mathrm{N}^\delta)^*\phi. \tag{101}$$

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